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A Neutrosophic Decision Analytics Model for the Sustainability Performance Assessment of Hydroelectric Power Plants

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ABSTRACT

Evaluating the sustainable performance of hydroelectric power plants is of paramount importance, considering their long-term impact on multiple dimensions. This paper proposes a new decision-making model that combines the type-2 neutrosophic logarithmic percentage change-driven objective weighting (LOPCOW) method with the combined compromise solution (CoCoSo) approach to evaluate the sustainable performance of hydroelectric power plants. The methodology incorporates type-2 neutrosophic sets to handle uncertainties efficiently. One of the main contributions of this study is the introduction of two new aggregation methods, which are designed using trigonometric t-norm and t-conorm functions to enhance decision-making under uncertainty. These operators enhance the accuracy of performance evaluation in complex and uncertain decision-making environments. To validate the hybrid model, a case study is conducted on hydroelectric power plants with an installed capacity from 10 to 50 megawatts in the Artvin province of Türkiye. The results of the model application demonstrated its effectiveness in assessing the sustainable performance of hydroelectric power plants. Furthermore, sensitivity analysis scenarios are employed to test the model's robustness, confirming its reliability. The model's ability to consider both qualitative and quantitative criteria, combined with its robustness, renders it a valuable tool for decision-makers in the field of sustainable energy.

1. Introduction

Hydroelectric power plants (HPPs) are renewable energy systems designed to generate electricity by converting the kinetic and potential energy of moving water into electrical energy through

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mechanical and electromagnetic processes [1]. HPPs are primarily established based on two different approaches [2]. The first approach involves facilities built on flowing water sources without a dam or storage reservoir [3]. These facilities aim to produce electricity from the natural flow of water and are known as "Run-of-River HPPs" [4]. The second approach includes facilities where water sources are obstructed by a dam or storage reservoir [5]. In these facilities, water is collected in a specific area, and then controlled flow from a designated point is used to generate electricity, and they are referred to as "Reservoir HPPs" [6]. In both types of facilities, turbines are rotated to harness the kinetic energy of water, driving electric generators to produce energy [7].

HPPs are established to meet the electricity demand, providing clean energy without air pollution or greenhouse gas emissions [8]. The operating costs of these plants are generally lower than other forms of energy generation, as they do not require fossil fuel consumption [9]. Additionally, Reservoir HPPs serve as a supportive element for managing water resources [10]. However, the construction of HPP projects and facility infrastructure involves significant investments. HPPs are widely used for electricity generation around the world, but they can have adverse environmental impacts, such as altering water ecosystems and inundating residential areas [11].

HPPs can vary in technical specifications. Reservoir HPPs can consist of various types of dams, including concrete dams, earth-fill dams, and steel dams [12]. Turbines can also differ in type and characteristics, with options like Kaplan, Francis, or Pelton turbines [13]. Generators are typically either induction generators or synchronous generators [14]. In terms of technical specifications, differences arise due to geographical conditions, water flow velocity, and other parameters, impacting the installed power capacity of HPPs [15]. Therefore, HPPs can differ in installed power capacity.

The primary goal of HPPs is electricity production. As a result, their performance assessment typically focuses on electricity generation capacity and the quantity of electricity generated [16]. Efforts are also made to evaluate their efficiency and effectiveness, as HPPs are long-term investments requiring control over the expected electricity production levels [17]. Additionally, the literature includes studies that employ artificial intelligence for maintenance planning of HPPs [18], investigations of optimal site selection for HPPs using multi-criteria decision-making (MCDM) methods [19], as well as research focusing on the optimal location selection of HPPs considering the water-energy-ecosystem nexus [20].

This research is conducted with the objective of assessing the sustainability performance of HPPs. However, the focus is not solely on electricity production but on evaluating the performance of HPPs in terms of sustainability. This approach recognizes that while HPPs contribute to energy production, they inevitably impact the local population, environment, social life, infrastructure, and other aspects. Moreover, the long-term sociological and environmental effects of HPPs play a significant role in their overall performance. Therefore, it is crucial to assess the sustainability performance of HPPs, considering their contributions to the national economy and other parameters.

A comprehensive model is introduced in this study to evaluate the sustainability performance of HPPs integrating both quantitative and qualitative indicators. Moreover, the model is designed based on expert evaluations for assessing qualitative criteria. MCDM methods, frequently employed in performance evaluation studies, are preferred in various fields, such as natural agribusinesses [21] and sustainable urban transportation [22], due to their suitability for analyzing numerous criteria and decision units. In the literature, various approaches have been proposed within the scope of neutrosophic logic, including the development of score functions [23], contributions to the circular economy [24], and hybrid applications with different MCDM methods [25]. In contrast, this study introduces a novel approach by developing aggregation operators and integrating them with MCDM techniques.

The model developed in this research, named “the type-2 neutrosophic number (T2NN)-logarithmic percentage change-driven objective weighting (LOPCOW)-combined compromise solution (CoCoSo) hybrid model”, treats the problem from the MCDM aspect. Within this model, experts evaluate HPPs based on the defined criteria. Neutrosophic sets, specifically T2NNs, are employed to reduce uncertainties in the decision problem. The LOPCOW method [26] is utilized for weighting criteria, and the CoCoSo method [27] is used for determining the sustainability performance and ranking of HPPs. The proposed hybrid model enables a comprehensive assessment by combining quantitative criteria informed by technical data with qualitative criteria informed by expert evaluations. Furthermore, two aggregation operators have been formulated for application in T2NNs, employing trigonometric *t-norm* and *t-conorm* operations to integrate fuzzy information effectively. Consequently, the model developed in this research makes a difference both theoretically and methodologically.

1.1 Aims and Contributions to the Study

This research primarily aims to construct a model for the systematic assessment of the sustainability performance of HPPs. This model utilizes the MCDM approach and leverages neutrosophic logic techniques, specifically T2NN sets. Additionally, two novel aggregation operators (namely, “the T2NN Trigonometric Weighted Arithmetic Mean (T2NNTWAM)” and “the T2NN Trigonometric Weighted Geometric Mean (T2NNTWGM)”) have been developed for integration into the proposed model. These operators are formulated using trigonometric *t-norm* and *t-conorm* operations, thereby enhancing the model’s versatility and computational effectiveness. Morever, a case study is conducted to illustrate the practical applicability of the proposed model within a real-world context.

This study makes significant contributions to the field of sustainable performance assessment, specifically within the context of HPPs:

i. *Model for assessing HPP sustainability* – The study proposes a comprehensive model that provides a holistic framework for evaluating the sustainability performance of HPPs. The model integrates both quantitative and qualitative criteria, encompassing technical measurements as well as expert assessments.
ii. *Successful model implementation* – The case study conducted in the research demonstrates the practical application for HPPs in the range of 10 to 50 megawatts (MW) installed capacity. The results of the case study confirm the efficacy of the proposed model.
iii. *Development of new aggregation operators* – The study introduces two novel aggregation operators, T2NNTWAM and T2NNTWGM, which are based on trigonometric *t*-norm and *t*-conorm operations. These expand the methodological toolkit for performance assessment and decision-making processes.
iv. *Robustness confirmation* – The sensitivity analysis scenarios conducted as part of the research serve to validate the robustness of the model, assuring its suitability for diverse conditions and criteria.
v. *Literature enrichment* – This research enriches academic literature by introducing new aggregation operators and contributing to the methodology of assessing the sustainable performance of HPPs from a multidimensional perspective.

In summary, this study advances the knowledge and methodology in the field of sustainable performance assessment for HPPs, offering a robust model and innovative aggregation operators, thereby benefiting researchers and practitioners in the energy sector.

This paper, which focuses on developing a model to assess the sustainable performance of HPPs, is organized into five main sections. Section 2 outlines the framework, offering a detailed explanation of the newly developed aggregation operators for T2NNs and the proposed T2NN-LOPCOW-CoCoSo hybrid model. In Section 3 presents a case study conducted in the Artvin province to evaluate the sustainability performance of HPPs. Section 4 discusses the results and their implications, including the application of sensitivity analysis scenarios and a comparative evaluation of the findings. Finally, Section 5 provides concluding remarks, highlighting the key outcomes.

2. Methodological Framework

This research is conducted in four methodological phases. The methodological framework representing these stages is illustrated in Fig. 1.

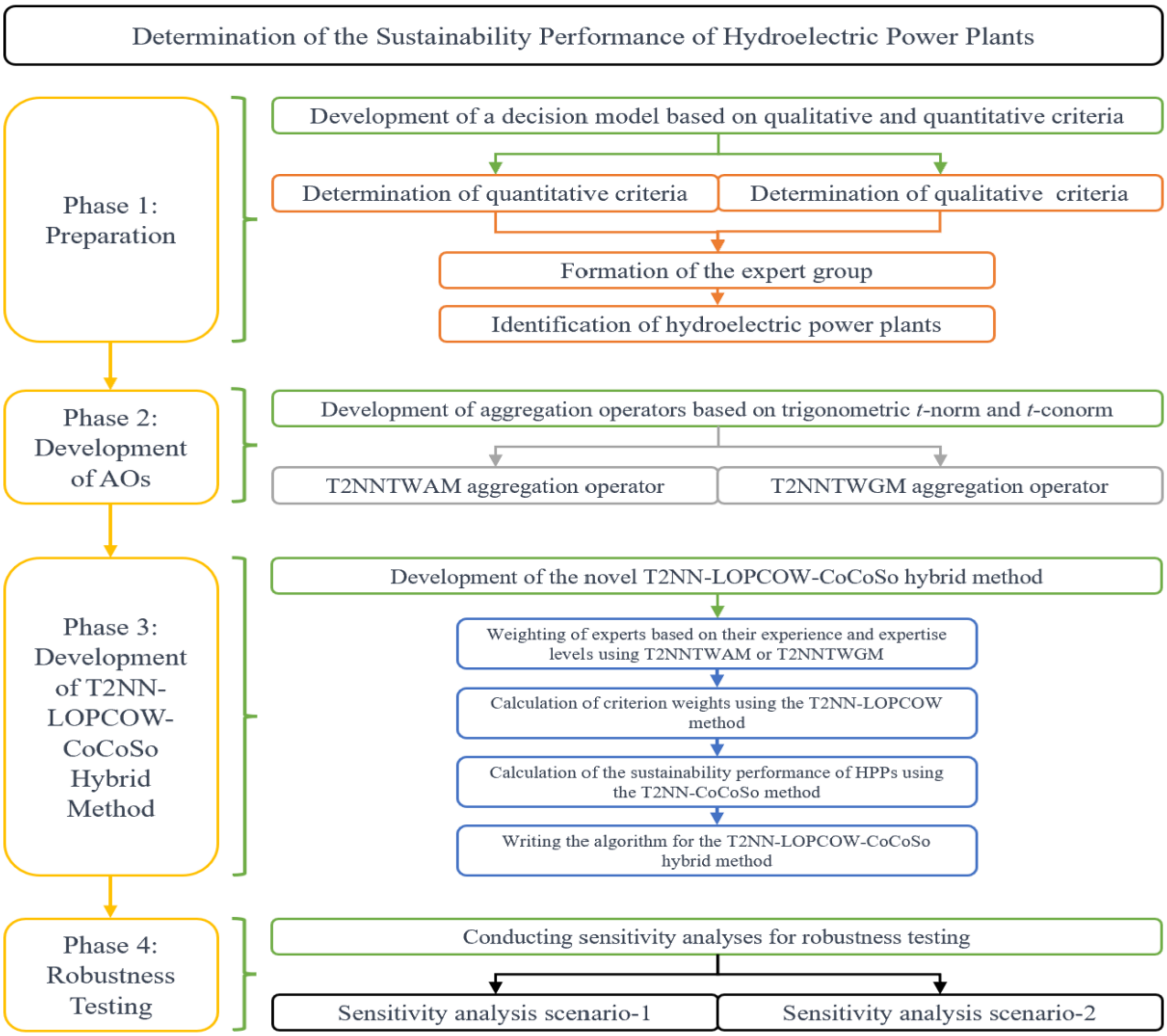

Fig. 1. Methodological framework

The initial phase is the preparation phase in which qualitative and quantitative criteria are identified, an expert group is formed, and the HPPs for which sustainability performances will be determined are identified. The second phase focuses on the development of aggregation operators. In this phase, T2NNTWAM and T2NNTWGM, based on trigonometric *t*-norm and *t*-conorm operations, are developed. The third phase is the model development phase. In this phase, the T2NN-LOPCOW-CoCoSo hybrid model is developed. The fourth phase is dedicated to robustness testing.

2.1 Determination of Decision Model

The process of evaluating HPPs in terms of sustainability performance is a delicate one. This process inherently involves both technical capabilities and environmental concerns. Therefore, it is expected that the decision matrices for sustainability performance assessment be carefully and accurately constructed. To meet this requirement, it is essential to accurately determine the three fundamental elements that constitute the decision matrix. These elements are experts, criteria, and HPPs. This stage is dedicated to the accurate identification of these three elements.

During the expert selection phase, it is essential to constitute a panel of experts capable of effectively representing both technical considerations and sustainability objectives. When determining the criteria, a combination of quantitative criteria that reflect technical specifications and qualitative criteria indicating sustainability expectations should be established. In the selection of HPPs, adhering to the principle of comparability, it is crucial to identify HPPs with similar characteristics and nearly identical capacities.

To meet these requirements, the precise formulation of the decision problem and the thorough examination of each of these steps are imperative. In this research, this stage, which pertains to the initiation of the research phase for the determination and comparison of HPPs' sustainability performance, is defined as the initial phase of the research and emphasized as the preparatory phase for the implementation stage.

2.2 The Formulation of Novel Aggregation Operators through Trigonometric Operations

This phase involves the enhancement and validation of the innovative T2NN operators, which constitute a core component of the model proposed in this research.

2.2.1 Trigonometric t-norm and t-conorm operations

This part provides an elucidation of trigonometric *t*-norm and *t*-conorm operations and their practical implementations in the context of T2NNs.

Definition 1. The trigonometric *t*-norm and *t*-conorm, expressed in Eq. (1) and Eq. (2) [29]:

$$\mathbb{T}(\mathbb{a},\mathbb{b}) = \frac{2}{\pi}\left(\cot^{-1}\left(\cot\left(\frac{\pi\mathbb{a}}{2}\right)+\cot\left(\frac{\pi\mathbb{b}}{2}\right)\right)\right), \tag{1}$$

$$\mathbb{T}^{*}(\mathbb{a},\mathbb{b}) = \frac{2}{\pi}\left(\tan^{-1}\left(\tan\left(\frac{\pi\mathbb{a}}{2}\right)+\tan\left(\frac{\pi\mathbb{b}}{2}\right)\right)\right), \tag{2}$$

where $\mathbb{T}(\mathbb{a},\mathbb{b})$: $[0,1]^2 \to [0,1]$ and $\mathbb{T}^{*}(\mathbb{a},\mathbb{b})$: $[0,1]^2 \to [0,1]$ for $\mathbb{a},\mathbb{b} \in [0,1]$.

Let us consider two T2NNs, denoted as $\tilde{\mathfrak{I}}_1$, and $\tilde{\mathfrak{I}}_2$, in universal set $\mathfrak{D}$, with their respective components articulated as: $\tilde{\mathfrak{I}}_1 = \langle\left(T_{T_{\tilde{\mathfrak{I}}_1}}(\hbar), T_{I_{\tilde{\mathfrak{I}}_1}}(\hbar), T_{F_{\tilde{\mathfrak{I}}_1}}(\hbar)\right), \left(I_{T_{\tilde{\mathfrak{I}}_1}}(\hbar), I_{I_{\tilde{\mathfrak{I}}_1}}(\hbar), I_{F_{\tilde{\mathfrak{I}}_1}}(\hbar)\right), \left(F_{T_{\tilde{\mathfrak{I}}_1}}(\hbar), F_{I_{\tilde{\mathfrak{I}}_1}}(\hbar), F_{F_{\tilde{\mathfrak{I}}_1}}(\hbar)\right)\rangle$, $\tilde{\mathfrak{I}}_2 = \langle\left(T_{T_{\tilde{\mathfrak{I}}_2}}(\hbar), T_{I_{\tilde{\mathfrak{I}}_2}}(\hbar), T_{F_{\tilde{\mathfrak{I}}_2}}(\hbar)\right), \left(I_{T_{\tilde{\mathfrak{I}}_2}}(\hbar), I_{I_{\tilde{\mathfrak{I}}_2}}(\hbar), I_{F_{\tilde{\mathfrak{I}}_2}}(\hbar)\right), \left(F_{T_{\tilde{\mathfrak{I}}_2}}(\hbar), F_{I_{\tilde{\mathfrak{I}}_2}}(\hbar), F_{F_{\tilde{\mathfrak{I}}_2}}(\hbar)\right)\rangle$. The trigonometric product $\left(\tilde{\mathfrak{I}}_1 \cap_{\mathbb{T},\mathbb{T}^*} \tilde{\mathfrak{I}}_2\right)$ and trigonometric sum $\left(\tilde{\mathfrak{I}}_1 \cup_{\mathbb{T},\mathbb{T}^*} \tilde{\mathfrak{I}}_2\right)$ of $\mathbb{T}$ and $\mathbb{T}^{*}$ are stated as Eq. (3) and Eq. (4), respectively:

$$\tilde{\mathfrak{J}}_{\mathbf{1}} \cap_{\mathbb{T},\mathbb{T}^*} \tilde{\mathfrak{J}}_{\mathbf{2}} = \left\{ \left\langle \begin{array}{l} \mathbb{T}\left(\left(\boldsymbol{T}_{\boldsymbol{T}_{\tilde{\mathfrak{J}}_1}}(\hbar), \boldsymbol{T}_{\boldsymbol{T}_{\tilde{\mathfrak{J}}_2}}(\hbar)\right), \left(\boldsymbol{T}_{\boldsymbol{I}_{\tilde{\mathfrak{J}}_1}}(\hbar), \boldsymbol{T}_{\boldsymbol{I}_{\tilde{\mathfrak{J}}_2}}(\hbar)\right), \left(\boldsymbol{T}_{\boldsymbol{F}_{\tilde{\mathfrak{J}}_1}}(\hbar), \boldsymbol{T}_{\boldsymbol{F}_{\tilde{\mathfrak{J}}_2}}(\hbar)\right)\right), \\ \mathbb{T}^*\left(\left(\boldsymbol{I}_{\boldsymbol{T}_{\tilde{\mathfrak{J}}_1}}(\hbar), \boldsymbol{I}_{\boldsymbol{T}_{\tilde{\mathfrak{J}}_2}}(\hbar)\right), \left(\boldsymbol{I}_{\boldsymbol{I}_{\tilde{\mathfrak{J}}_1}}(\hbar), \boldsymbol{I}_{\boldsymbol{I}_{\tilde{\mathfrak{J}}_2}}(\hbar)\right), \left(\boldsymbol{I}_{\boldsymbol{F}_{\tilde{\mathfrak{J}}_1}}(\hbar), \boldsymbol{I}_{\boldsymbol{F}_{\tilde{\mathfrak{J}}_2}}(\hbar)\right)\right), \\ \mathbb{T}^*\left(\left(\boldsymbol{F}_{\boldsymbol{T}_{\tilde{\mathfrak{J}}_1}}(\hbar), \boldsymbol{F}_{\boldsymbol{T}_{\tilde{\mathfrak{J}}_2}}(\hbar)\right), \left(\boldsymbol{F}_{\boldsymbol{I}_{\tilde{\mathfrak{J}}_1}}(\hbar), \boldsymbol{F}_{\boldsymbol{I}_{\tilde{\mathfrak{J}}_2}}(\hbar)\right), \left(\boldsymbol{F}_{\boldsymbol{F}_{\tilde{\mathfrak{J}}_1}}(\hbar), \boldsymbol{F}_{\boldsymbol{F}_{\tilde{\mathfrak{J}}_2}}(\hbar)\right)\right) \end{array} \right\rangle \right\}, \tag{3}$$

$$\tilde{\mathfrak{J}}_{\mathbf{1}} \cup_{\mathbb{T},\mathbb{T}^*} \tilde{\mathfrak{J}}_{\mathbf{2}} = \left\{ \left\langle \begin{array}{l} \mathbb{T}^*\left(\left(\boldsymbol{T}_{\boldsymbol{T}_{\tilde{\mathfrak{J}}_1}}(\hbar), \boldsymbol{T}_{\boldsymbol{T}_{\tilde{\mathfrak{J}}_2}}(\hbar)\right), \left(\boldsymbol{T}_{\boldsymbol{I}_{\tilde{\mathfrak{J}}_1}}(\hbar), \boldsymbol{T}_{\boldsymbol{I}_{\tilde{\mathfrak{J}}_2}}(\hbar)\right), \left(\boldsymbol{T}_{\boldsymbol{F}_{\tilde{\mathfrak{J}}_1}}(\hbar), \boldsymbol{T}_{\boldsymbol{F}_{\tilde{\mathfrak{J}}_2}}(\hbar)\right)\right), \\ \mathbb{T}\left(\left(\boldsymbol{I}_{\boldsymbol{T}_{\tilde{\mathfrak{J}}_1}}(\hbar), \boldsymbol{I}_{\boldsymbol{T}_{\tilde{\mathfrak{J}}_2}}(\hbar)\right), \left(\boldsymbol{I}_{\boldsymbol{I}_{\tilde{\mathfrak{J}}_1}}(\hbar), \boldsymbol{I}_{\boldsymbol{I}_{\tilde{\mathfrak{J}}_2}}(\hbar)\right), \left(\boldsymbol{I}_{\boldsymbol{F}_{\tilde{\mathfrak{J}}_1}}(\hbar), \boldsymbol{I}_{\boldsymbol{F}_{\tilde{\mathfrak{J}}_2}}(\hbar)\right)\right), \\ \mathbb{T}\left(\left(\boldsymbol{F}_{\boldsymbol{T}_{\tilde{\mathfrak{J}}_1}}(\hbar), \boldsymbol{F}_{\boldsymbol{T}_{\tilde{\mathfrak{J}}_2}}(\hbar)\right), \left(\boldsymbol{F}_{\boldsymbol{I}_{\tilde{\mathfrak{J}}_1}}(\hbar), \boldsymbol{F}_{\boldsymbol{I}_{\tilde{\mathfrak{J}}_2}}(\hbar)\right), \left(\boldsymbol{F}_{\boldsymbol{F}_{\tilde{\mathfrak{J}}_1}}(\hbar), \boldsymbol{F}_{\boldsymbol{F}_{\tilde{\mathfrak{J}}_2}}(\hbar)\right)\right) \end{array} \right\rangle \right\} \tag{4}$$

Definition 2. Consider two T2NNs as $\tilde{\mathfrak{J}}_1$, and $\tilde{\mathfrak{J}}_2$, identified as set $\mathfrak{D}$, with their respective components as $\tilde{\mathfrak{J}}_1 = \left\langle \left(T_{T_{\tilde{\mathfrak{J}}_1}}(\hbar), T_{I_{\tilde{\mathfrak{J}}_1}}(\hbar), T_{F_{\tilde{\mathfrak{J}}_1}}(\hbar)\right), \left(I_{T_{\tilde{\mathfrak{J}}_1}}(\hbar), I_{I_{\tilde{\mathfrak{J}}_1}}(\hbar), I_{F_{\tilde{\mathfrak{J}}_1}}(\hbar)\right), \left(F_{T_{\tilde{\mathfrak{J}}_1}}(\hbar), F_{I_{\tilde{\mathfrak{J}}_1}}(\hbar), F_{F_{\tilde{\mathfrak{J}}_1}}(\hbar)\right) \right\rangle$, $\tilde{\mathfrak{J}}_2 = \left\langle \left(T_{T_{\tilde{\mathfrak{J}}_2}}(\hbar), T_{I_{\tilde{\mathfrak{J}}_2}}(\hbar), T_{F_{\tilde{\mathfrak{J}}_2}}(\hbar)\right), \left(I_{T_{\tilde{\mathfrak{J}}_2}}(\hbar), I_{I_{\tilde{\mathfrak{J}}_2}}(\hbar), I_{F_{\tilde{\mathfrak{J}}_2}}(\hbar)\right), \left(F_{T_{\tilde{\mathfrak{J}}_2}}(\hbar), F_{I_{\tilde{\mathfrak{J}}_2}}(\hbar), F_{F_{\tilde{\mathfrak{J}}_2}}(\hbar)\right) \right\rangle$. The procedures regulating the interaction between two T2NNs, utilizing the trigonometric *t*-norm and trigonometric *t*-conorm, are elucidated below, under the condition that $\mathfrak{s} > 0$:

(i) $$\tilde{\mathfrak{J}}_1 \otimes \tilde{\mathfrak{J}}_2 = \left\{ \begin{array}{l} \left(\frac{2}{\pi}\left(cot^{-1}\left(cot\left(\frac{\pi\left(T_{T_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right) + cot\left(\frac{\pi\left(T_{T_{\tilde{\mathfrak{J}}_2}}(\hbar)\right)}{2}\right)\right)\right), \frac{2}{\pi}\left(cot^{-1}\left(cot\left(\frac{\pi\left(T_{I_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right) + cot\left(\frac{\pi\left(T_{I_{\tilde{\mathfrak{J}}_2}}(\hbar)\right)}{2}\right)\right)\right), \frac{2}{\pi}\left(cot^{-1}\left(cot\left(\frac{\pi\left(T_{F_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right) + cot\left(\frac{\pi\left(T_{F_{\tilde{\mathfrak{J}}_2}}(\hbar)\right)}{2}\right)\right)\right)\right), \\ \left(\frac{2}{\pi}\left(tan^{-1}\left(tan\left(\frac{\pi\left(I_{T_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right) + tan\left(\frac{\pi\left(I_{T_{\tilde{\mathfrak{J}}_2}}(\hbar)\right)}{2}\right)\right)\right), \frac{2}{\pi}\left(tan^{-1}\left(tan\left(\frac{\pi\left(I_{I_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right) + tan\left(\frac{\pi\left(I_{I_{\tilde{\mathfrak{J}}_2}}(\hbar)\right)}{2}\right)\right)\right), \frac{2}{\pi}\left(tan^{-1}\left(tan\left(\frac{\pi\left(I_{F_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right) + tan\left(\frac{\pi\left(I_{F_{\tilde{\mathfrak{J}}_2}}(\hbar)\right)}{2}\right)\right)\right)\right), \\ \left(\frac{2}{\pi}\left(tan^{-1}\left(tan\left(\frac{\pi\left(F_{T_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right) + tan\left(\frac{\pi\left(F_{T_{\tilde{\mathfrak{J}}_2}}(\hbar)\right)}{2}\right)\right)\right), \frac{2}{\pi}\left(tan^{-1}\left(tan\left(\frac{\pi\left(F_{I_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right) + tan\left(\frac{\pi\left(F_{I_{\tilde{\mathfrak{J}}_2}}(\hbar)\right)}{2}\right)\right)\right), \frac{2}{\pi}\left(tan^{-1}\left(tan\left(\frac{\pi\left(F_{F_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right) + tan\left(\frac{\pi\left(F_{F_{\tilde{\mathfrak{J}}_2}}(\hbar)\right)}{2}\right)\right)\right)\right) \end{array} \right\},$$

(ii) $$\tilde{\mathfrak{J}}_1 \oplus \tilde{\mathfrak{J}}_2 = \left\{ \begin{array}{l} \left(\frac{2}{\pi}\left(tan^{-1}\left(tan\left(\frac{\pi\left(T_{T_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right) + tan\left(\frac{\pi\left(T_{T_{\tilde{\mathfrak{J}}_2}}(\hbar)\right)}{2}\right)\right)\right), \frac{2}{\pi}\left(tan^{-1}\left(tan\left(\frac{\pi\left(T_{I_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right) + tan\left(\frac{\pi\left(T_{I_{\tilde{\mathfrak{J}}_2}}(\hbar)\right)}{2}\right)\right)\right), \frac{2}{\pi}\left(tan^{-1}\left(tan\left(\frac{\pi\left(T_{F_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right) + tan\left(\frac{\pi\left(T_{F_{\tilde{\mathfrak{J}}_2}}(\hbar)\right)}{2}\right)\right)\right)\right), \\ \left(\frac{2}{\pi}\left(cot^{-1}\left(cot\left(\frac{\pi\left(I_{T_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right) + cot\left(\frac{\pi\left(I_{T_{\tilde{\mathfrak{J}}_2}}(\hbar)\right)}{2}\right)\right)\right), \frac{2}{\pi}\left(cot^{-1}\left(cot\left(\frac{\pi\left(I_{I_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right) + cot\left(\frac{\pi\left(I_{I_{\tilde{\mathfrak{J}}_2}}(\hbar)\right)}{2}\right)\right)\right), \frac{2}{\pi}\left(cot^{-1}\left(cot\left(\frac{\pi\left(I_{F_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right) + cot\left(\frac{\pi\left(I_{F_{\tilde{\mathfrak{J}}_2}}(\hbar)\right)}{2}\right)\right)\right)\right), \\ \left(\frac{2}{\pi}\left(cot^{-1}\left(cot\left(\frac{\pi\left(F_{T_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right) + cot\left(\frac{\pi\left(F_{T_{\tilde{\mathfrak{J}}_2}}(\hbar)\right)}{2}\right)\right)\right), \frac{2}{\pi}\left(cot^{-1}\left(cot\left(\frac{\pi\left(F_{I_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right) + cot\left(\frac{\pi\left(F_{I_{\tilde{\mathfrak{J}}_2}}(\hbar)\right)}{2}\right)\right)\right), \frac{2}{\pi}\left(cot^{-1}\left(cot\left(\frac{\pi\left(F_{F_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right) + cot\left(\frac{\pi\left(F_{F_{\tilde{\mathfrak{J}}_2}}(\hbar)\right)}{2}\right)\right)\right)\right) \end{array} \right\},$$

(iii) $$\mathfrak{s}\tilde{\mathfrak{J}}_1 = \left\{ \begin{array}{l} \left(\frac{2}{\pi}\left(tan^{-1}\left(\mathfrak{s}\left(tan\left(\frac{\pi\left(T_{T_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right)\right)\right)\right), \frac{2}{\pi}\left(tan^{-1}\left(\mathfrak{s}\left(tan\left(\frac{\pi\left(T_{I_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right)\right)\right)\right), \frac{2}{\pi}\left(tan^{-1}\left(\mathfrak{s}\left(tan\left(\frac{\pi\left(T_{F_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right)\right)\right)\right)\right), \\ \left(\frac{2}{\pi}\left(cot^{-1}\left(\mathfrak{s}\left(cot\left(\frac{\pi\left(I_{T_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right)\right)\right)\right), \frac{2}{\pi}\left(cot^{-1}\left(\mathfrak{s}\left(cot\left(\frac{\pi\left(I_{I_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right)\right)\right)\right), \frac{2}{\pi}\left(cot^{-1}\left(\mathfrak{s}\left(cot\left(\frac{\pi\left(I_{F_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right)\right)\right)\right)\right), \\ \left(\frac{2}{\pi}\left(cot^{-1}\left(\mathfrak{s}\left(cot\left(\frac{\pi\left(F_{T_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right)\right)\right)\right), \frac{2}{\pi}\left(cot^{-1}\left(\mathfrak{s}\left(cot\left(\frac{\pi\left(F_{I_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right)\right)\right)\right), \frac{2}{\pi}\left(cot^{-1}\left(\mathfrak{s}\left(cot\left(\frac{\pi\left(F_{F_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right)\right)\right)\right)\right) \end{array} \right\},$$

(iv) $$\tilde{\mathfrak{J}}_1^{\mathfrak{s}} = \left\{ \begin{array}{l} \left(\frac{2}{\pi}\left(cot^{-1}\left(\mathfrak{s}\left(cot\left(\frac{\pi\left(T_{T_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right)\right)\right)\right), \frac{2}{\pi}\left(cot^{-1}\left(\mathfrak{s}\left(cot\left(\frac{\pi\left(T_{I_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right)\right)\right)\right), \frac{2}{\pi}\left(cot^{-1}\left(\mathfrak{s}\left(cot\left(\frac{\pi\left(T_{F_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right)\right)\right)\right)\right), \\ \left(\frac{2}{\pi}\left(tan^{-1}\left(\mathfrak{s}\left(tan\left(\frac{\pi\left(I_{T_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right)\right)\right)\right), \frac{2}{\pi}\left(tan^{-1}\left(\mathfrak{s}\left(tan\left(\frac{\pi\left(I_{I_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right)\right)\right)\right), \frac{2}{\pi}\left(tan^{-1}\left(\mathfrak{s}\left(tan\left(\frac{\pi\left(I_{F_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right)\right)\right)\right)\right), \\ \left(\frac{2}{\pi}\left(tan^{-1}\left(\mathfrak{s}\left(tan\left(\frac{\pi\left(F_{T_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right)\right)\right)\right), \frac{2}{\pi}\left(tan^{-1}\left(\mathfrak{s}\left(tan\left(\frac{\pi\left(F_{I_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right)\right)\right)\right), \frac{2}{\pi}\left(tan^{-1}\left(\mathfrak{s}\left(tan\left(\frac{\pi\left(F_{F_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)}{2}\right)\right)\right)\right)\right) \end{array} \right\}.$$

Theorem 1. Consider two T2NNs as $\tilde{\mathfrak{J}}_1$, and $\tilde{\mathfrak{J}}_2$, identified as set $\mathfrak{D}$, with their respective components as: $\tilde{\mathfrak{J}}_1 = \langle\left(T_{T_{\tilde{\mathfrak{J}}_1}}(\hbar), T_{I_{\tilde{\mathfrak{J}}_1}}(\hbar), T_{F_{\tilde{\mathfrak{J}}_1}}(\hbar)\right), \left(I_{T_{\tilde{\mathfrak{J}}_1}}(\hbar), I_{I_{\tilde{\mathfrak{J}}_1}}(\hbar), I_{F_{\tilde{\mathfrak{J}}_1}}(\hbar)\right), \left(F_{T_{\tilde{\mathfrak{J}}_1}}(\hbar), F_{I_{\tilde{\mathfrak{J}}_1}}(\hbar), F_{F_{\tilde{\mathfrak{J}}_1}}(\hbar)\right)\rangle$, $\tilde{\mathfrak{J}}_2 = \langle\left(T_{T_{\tilde{\mathfrak{J}}_2}}(\hbar), T_{I_{\tilde{\mathfrak{J}}_2}}(\hbar), T_{F_{\tilde{\mathfrak{J}}_2}}(\hbar)\right), \left(I_{T_{\tilde{\mathfrak{J}}_2}}(\hbar), I_{I_{\tilde{\mathfrak{J}}_2}}(\hbar), I_{F_{\tilde{\mathfrak{J}}_2}}(\hbar)\right), \left(F_{T_{\tilde{\mathfrak{J}}_2}}(\hbar), F_{I_{\tilde{\mathfrak{J}}_2}}(\hbar), F_{F_{\tilde{\mathfrak{J}}_2}}(\hbar)\right)\rangle$. In the case of the trigonometric *t*-norm and trigonometric *t*-conorm, the methods entail the collaborative interaction between two T2NNs, which is described as follows:

i. $\tilde{\mathfrak{J}}_1 \oplus \tilde{\mathfrak{J}}_2 = \tilde{\mathfrak{J}}_2 \oplus \tilde{\mathfrak{J}}_1$,
ii. $\tilde{\mathfrak{J}}_1 \otimes \tilde{\mathfrak{J}}_2 = \tilde{\mathfrak{J}}_2 \otimes \tilde{\mathfrak{J}}_1$,
iii. $\mathfrak{s}\left(\tilde{\mathfrak{J}}_1 \oplus \tilde{\mathfrak{J}}_2\right) = \mathfrak{s}\tilde{\mathfrak{J}}_1 \oplus \mathfrak{s}\tilde{\mathfrak{J}}_2$ for $\mathfrak{s} > 0$,
iv. $\left(\tilde{\mathfrak{J}}_1 \otimes \tilde{\mathfrak{J}}_2\right)^{\mathfrak{s}} = \tilde{\mathfrak{J}}_1^{\mathfrak{s}} \otimes \tilde{\mathfrak{J}}_2^{\mathfrak{s}}$ for $\mathfrak{s} > 0$,
v. $\mathfrak{s}_1\tilde{\mathfrak{J}} \oplus \mathfrak{s}_2\tilde{\mathfrak{J}} = (\mathfrak{s}_1 + \mathfrak{s}_2)\tilde{\mathfrak{J}}$ for $\mathfrak{s}_1, \mathfrak{s}_2 > 0$,
vi. $\tilde{\mathfrak{J}}^{\mathfrak{s}_1} \otimes \tilde{\mathfrak{J}}^{\mathfrak{s}_2} = \tilde{\mathfrak{J}}^{(\mathfrak{s}_1 + \mathfrak{s}_2)}$ for $\mathfrak{s}_1, \mathfrak{s}_2 > 0$.

Proof: Proof is clear.

2.2.2 T2NN trigonometric weighted arithmetic mean aggregation operator

In this sub-section, T2NNTWAM is presented.

Definition 3. Let us consider $\tilde{\mathfrak{J}}_{\mathfrak{g}}$, denoted as $\tilde{\mathfrak{J}}_{\mathfrak{g}} = \langle\left(T_{T_{\tilde{\mathfrak{J}}_{\mathfrak{g}}}}(\hbar), T_{I_{\tilde{\mathfrak{J}}_{\mathfrak{g}}}}(\hbar), T_{F_{\tilde{\mathfrak{J}}_{\mathfrak{g}}}}(\hbar)\right), \left(I_{T_{\tilde{\mathfrak{J}}_{\mathfrak{g}}}}(\hbar), I_{I_{\tilde{\mathfrak{J}}_{\mathfrak{g}}}}(\hbar), I_{F_{\tilde{\mathfrak{J}}_{\mathfrak{g}}}}(\hbar)\right), \left(F_{T_{\tilde{\mathfrak{J}}_{\mathfrak{g}}}}(\hbar), F_{I_{\tilde{\mathfrak{J}}_{\mathfrak{g}}}}(\hbar), F_{F_{\tilde{\mathfrak{J}}_{\mathfrak{g}}}}(\hbar)\right)\rangle$, which represents a T2NN. Furthermore, we have the associated vector $\mathfrak{s} = \left(\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_{\mathfrak{h}}\right)$ for $\mathfrak{s}_{\mathfrak{g}}\, \epsilon\, [0,1]$ and $\sum_{\mathfrak{g}=1}^{\mathfrak{h}} \mathfrak{s}_{\mathfrak{g}} = 1$. Here, the T2NNTWAM operator is defined by Eq. (5):

$$\boldsymbol{T2NNYWAM}\left(\tilde{\mathfrak{J}}_{\mathbf{1}}, \tilde{\mathfrak{J}}_{\mathbf{2}}, \dots, \tilde{\mathfrak{J}}_{\mathfrak{h}}\right) = \oplus_{\mathfrak{g}=\mathbf{1}}^{\mathfrak{h}}\, \mathfrak{s}_{\mathfrak{g}}\tilde{\mathfrak{J}}_{\mathfrak{g}}. \tag{5}$$

Theorem 2: Let us consider $\tilde{\mathfrak{J}}_{\mathfrak{g}}$, denoted as $\tilde{\mathfrak{J}}_{\mathfrak{g}} = \langle\left(T_{T_{\tilde{\mathfrak{J}}_{\mathfrak{g}}}}(\hbar), T_{I_{\tilde{\mathfrak{J}}_{\mathfrak{g}}}}(\hbar), T_{F_{\tilde{\mathfrak{J}}_{\mathfrak{g}}}}(\hbar)\right), \left(I_{T_{\tilde{\mathfrak{J}}_{\mathfrak{g}}}}(\hbar), I_{I_{\tilde{\mathfrak{J}}_{\mathfrak{g}}}}(\hbar), I_{F_{\tilde{\mathfrak{J}}_{\mathfrak{g}}}}(\hbar)\right), \left(F_{T_{\tilde{\mathfrak{J}}_{\mathfrak{g}}}}(\hbar), F_{I_{\tilde{\mathfrak{J}}_{\mathfrak{g}}}}(\hbar), F_{F_{\tilde{\mathfrak{J}}_{\mathfrak{g}}}}(\hbar)\right)\rangle$, which represents a T2NN. Furthermore, we have the associated weight vector $\mathfrak{s} = \left(\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_{\mathfrak{h}}\right)$ for $\mathfrak{s}_{\mathfrak{g}}\, \epsilon\, [0,1]$ and $\sum_{\mathfrak{g}=1}^{\mathfrak{h}} \mathfrak{s}_{\mathfrak{g}} = 1$ and $\pi = 3.14$. Then, the T2NNTWAM aggregation operator theorem is stated as Eq. (6):

$$\boldsymbol{T2NNTWAM}(\tilde{\mathfrak{Z}}_1,\tilde{\mathfrak{Z}}_2,\dots,\tilde{\mathfrak{Z}}_{\mathfrak{h}}) = \oplus_{g=1}^{\mathfrak{h}} \mathfrak{s}_g \tilde{\mathfrak{Z}}_g =$$

$$\left\{\begin{array}{l}
\left(\left(\frac{2}{\pi}\left(tan^{-1}\left(\sum_{g=1}^{\mathfrak{h}}\left(\mathfrak{s}_g\left(tan\left(\frac{\pi\left(T_{T_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right)}{2}\right)\right)\right)\right)\right),\frac{2}{\pi}\left(tan^{-1}\left(\sum_{g=1}^{\mathfrak{h}}\left(\mathfrak{s}_g\left(tan\left(\frac{\pi\left(T_{I_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right)}{2}\right)\right)\right)\right)\right),\frac{2}{\pi}\left(tan^{-1}\left(\sum_{g=1}^{\mathfrak{h}}\left(\mathfrak{s}_g\left(tan\left(\frac{\pi\left(T_{F_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right)}{2}\right)\right)\right)\right)\right)\right),\right.\\
\left(\frac{2}{\pi}\left(cot^{-1}\left(\sum_{g=1}^{\mathfrak{h}}\left(\mathfrak{s}_g\left(cot\left(\frac{\pi\left(I_{T_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right)}{2}\right)\right)\right)\right)\right),\frac{2}{\pi}\left(cot^{-1}\left(\sum_{g=1}^{\mathfrak{h}}\left(\mathfrak{s}_g\left(cot\left(\frac{\pi\left(I_{I_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right)}{2}\right)\right)\right)\right)\right),\frac{2}{\pi}\left(cot^{-1}\left(\sum_{g=1}^{\mathfrak{h}}\left(\mathfrak{s}_g\left(cot\left(\frac{\pi\left(I_{F_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right)}{2}\right)\right)\right)\right)\right)\right),\\
\left.\left(\frac{2}{\pi}\left(cot^{-1}\left(\sum_{g=1}^{\mathfrak{h}}\left(\mathfrak{s}_g\left(cot\left(\frac{\pi\left(F_{T_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right)}{2}\right)\right)\right)\right)\right),\frac{2}{\pi}\left(cot^{-1}\left(\sum_{g=1}^{\mathfrak{h}}\left(\mathfrak{s}_g\left(cot\left(\frac{\pi\left(F_{I_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right)}{2}\right)\right)\right)\right)\right),\frac{2}{\pi}\left(cot^{-1}\left(\sum_{g=1}^{\mathfrak{h}}\left(\mathfrak{s}_g\left(cot\left(\frac{\pi\left(F_{F_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right)}{2}\right)\right)\right)\right)\right)\right)\right)
\end{array}\right\} \quad (6)$$

Proof: Proof is shown in Proof-1 in Appendix B.

Theorem 3: The T2NNTWAM operator complies with the following properties:

(i) *Monotonicity*: Consider that $\tilde{\mathfrak{Z}}_g = \langle\left(T_{T_{\tilde{\mathfrak{Z}}_g}}(\hbar), T_{I_{\tilde{\mathfrak{Z}}_g}}(\hbar), T_{F_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right), \left(I_{T_{\tilde{\mathfrak{Z}}_g}}(\hbar), I_{I_{\tilde{\mathfrak{Z}}_g}}(\hbar), I_{F_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right), \left(F_{T_{\tilde{\mathfrak{Z}}_g}}(\hbar), F_{I_{\tilde{\mathfrak{Z}}_g}}(\hbar), F_{F_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right)\rangle$ and $\tilde{\mathfrak{Z}}_g^{\ *} = \langle\left(T_{T_{\tilde{\mathfrak{Z}}_g^*}}(\hbar), T_{I_{\tilde{\mathfrak{Z}}_g^*}}(\hbar), T_{F_{\tilde{\mathfrak{Z}}_g^*}}(\hbar)\right), \left(I_{T_{\tilde{\mathfrak{Z}}_g^*}}(\hbar), I_{I_{\tilde{\mathfrak{Z}}_g^*}}(\hbar), I_{F_{\tilde{\mathfrak{Z}}_g^*}}(\hbar)\right), \left(F_{T_{\tilde{\mathfrak{Z}}_g^*}}(\hbar), F_{I_{\tilde{\mathfrak{Z}}_g^*}}(\hbar), F_{F_{\tilde{\mathfrak{Z}}_g^*}}(\hbar)\right)\rangle$ are two T2NNs over the universe $\mathfrak{D}$. If $T_{T_{\tilde{\mathfrak{Z}}_g}}(\hbar) \ge T_{T_{\tilde{\mathfrak{Z}}_g^*}}(\hbar), T_{I_{\tilde{\mathfrak{Z}}_g}}(\hbar) \ge T_{I_{\tilde{\mathfrak{Z}}_g^*}}(\hbar), T_{F_{\tilde{\mathfrak{Z}}_g}}(\hbar) \ge T_{F_{\tilde{\mathfrak{Z}}_g^*}}(\hbar)$ and $I_{T_{\tilde{\mathfrak{Z}}_g}}(\hbar) \le I_{T_{\tilde{\mathfrak{Z}}_g^*}}(\hbar), I_{I_{\tilde{\mathfrak{Z}}_g}}(\hbar) \le I_{I_{\tilde{\mathfrak{Z}}_g^*}}(\hbar), I_{F_{\tilde{\mathfrak{Z}}_g}}(\hbar) \le I_{F_{\tilde{\mathfrak{Z}}_g^*}}(\hbar)$ and $F_{T_{\tilde{\mathfrak{Z}}_g}}(\hbar) \ge F_{T_{\tilde{\mathfrak{Z}}_g^*}}(\hbar), F_{I_{\tilde{\mathfrak{Z}}_g}}(\hbar) \le F_{I_{\tilde{\mathfrak{Z}}_g^*}}(\hbar), F_{F_{\tilde{\mathfrak{Z}}_g}}(\hbar) \le F_{F_{\tilde{\mathfrak{Z}}_g^*}}(\hbar)$, then $T2NNTWAM(\tilde{\mathfrak{Z}}_1,\tilde{\mathfrak{Z}}_2,\dots,\tilde{\mathfrak{Z}}_{\mathfrak{h}}) \ge T2NNTWAM(\tilde{\mathfrak{Z}}_1^{\ *},\tilde{\mathfrak{Z}}_2^{\ *},\dots,\tilde{\mathfrak{Z}}_{\mathfrak{h}}^{\ *})$.

Proof: Proof is shown in Proof-2 in Appendix B.

(ii) *Idempotency*: If $\tilde{\mathfrak{Z}}_g = \tilde{\mathfrak{Z}}$, $T_{T_{\tilde{\mathfrak{Z}}}}(\hbar) = T_{T_{\tilde{\mathfrak{Z}}_g}}(\hbar), T_{I_{\tilde{\mathfrak{Z}}}}(\hbar) = T_{I_{\tilde{\mathfrak{Z}}_g}}(\hbar)$, $T_{F_{\tilde{\mathfrak{Z}}}}(\hbar) = T_{F_{\tilde{\mathfrak{Z}}_g}}(\hbar)$, and $I_{T_{\tilde{\mathfrak{Z}}}}(\hbar) = I_{T_{\tilde{\mathfrak{Z}}_g}}(\hbar), I_{I_{\tilde{\mathfrak{Z}}}}(\hbar) = I_{I_{\tilde{\mathfrak{Z}}_g}}(\hbar)$, $I_{F_{\tilde{\mathfrak{Z}}}}(\hbar) = I_{F_{\tilde{\mathfrak{Z}}_g}}(\hbar)$, and $F_{T_{\tilde{\mathfrak{Z}}}}(\hbar) = F_{T_{\tilde{\mathfrak{Z}}_g}}(\hbar), F_{I_{\tilde{\mathfrak{Z}}}}(\hbar) = F_{I_{\tilde{\mathfrak{Z}}_g}}(\hbar)$, $F_{F_{\tilde{\mathfrak{Z}}}}(\hbar) = F_{F_{\tilde{\mathfrak{Z}}_g}}(\hbar)$, then $T2NNTWAM(\tilde{\mathfrak{Z}}_1,\tilde{\mathfrak{Z}}_2,\dots,\tilde{\mathfrak{Z}}_{\mathfrak{h}}) = \tilde{\mathfrak{Z}}$.

Proof: Proof is presented in Proof-3 in Appendix B.

(iii) *Boundedness*: Suppose the maximum and minimum T2NNs are as follows:

$\tilde{\mathfrak{Z}}_{max} = \langle\left(max_g\left(T_{T_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right), max_g\left(T_{I_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right) max_g\left(T_{F_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right)\right),$
$\left(min_g\left(I_{T_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right), min_g\left(I_{I_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right) min_g\left(I_{F_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right)\right), \left(min_g\left(F_{T_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right), min_g\left(F_{I_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right) min_g\left(F_{F_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right)\right)\rangle,$

$\tilde{\mathfrak{Z}}_{min} = \langle\left(min_g\left(T_{T_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right), min_g\left(T_{I_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right) min_g\left(T_{F_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right)\right),$
$\left(max_g\left(I_{T_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right), max_g\left(I_{I_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right) max_g\left(I_{F_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right)\right), \left(max_g\left(F_{T_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right), max_g\left(F_{I_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right) max_g\left(F_{F_{\tilde{\mathfrak{Z}}_g}}(\hbar)\right)\right)\rangle,$

then, $\tilde{\mathfrak{Z}}_{max} \le \mathrm{T2}NNTWAM(\tilde{\mathfrak{Z}}_1,\tilde{\mathfrak{Z}}_2,\dots,\tilde{\mathfrak{Z}}_{\mathfrak{h}}) \le \tilde{\mathfrak{Z}}_{min}$ can hold.

Proof: Proof is clear.

Numerical example: The weights of the three decision-makers $(\mathfrak{s}_g)$ are 0.3, 0.3, and 0.4, respectively. The T2NN assigned by the decision-makers are presented as follows: $\tilde{\mathfrak{Z}}_1 = \langle(0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05)\rangle$, $\tilde{\mathfrak{Z}}_1 = \langle(0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05)\rangle$ and $\tilde{\mathfrak{Z}}_3 = \langle(0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20)\rangle$. The procedures of the T2NNTWAM aggregation operator are as follows:

$\frac{2}{\pi}\left(tan^{-1}\left(0.3\left(tan\left(\frac{\pi(0.95)}{2}\right)\right)+0.3\left(tan\left(\frac{\pi(0.95)}{2}\right)\right)+0.4\left(tan\left(\frac{\pi(0.70)}{2}\right)\right)\right)\right)=0.9244,$

$\frac{2}{\pi}\left(tan^{-1}\left(0.3\left(tan\left(\frac{\pi(0.90)}{2}\right)\right)+0.3\left(tan\left(\frac{\pi(0.90)}{2}\right)\right)+0.4\left(tan\left(\frac{\pi(0.75)}{2}\right)\right)\right)\right)=0.8679,$

$\frac{2}{\pi}\left(tan^{-1}\left(0.3\left(tan\left(\frac{\pi(0.95)}{2}\right)\right)+0.3\left(tan\left(\frac{\pi(0.95)}{2}\right)\right)+0.4\left(tan\left(\frac{\pi(0.80)}{2}\right)\right)\right)\right)=0.9283,$

$\frac{2}{\pi}\left(cot^{-1}\left(0.3\left(cot\left(\frac{\pi(0.10)}{2}\right)\right)+0.3\left(cot\left(\frac{\pi(0.10)}{2}\right)\right)+0.4\left(cot\left(\frac{\pi(0.15)}{2}\right)\right)\right)\right)=0.1154,$

$\frac{2}{\pi}\left(cot^{-1}\left(0.3\left(cot\left(\frac{\pi(0.10)}{2}\right)\right)+0.3\left(cot\left(\frac{\pi(0.10)}{2}\right)\right)+0.4\left(cot\left(\frac{\pi(0.20)}{2}\right)\right)\right)\right)=0.1252,$

$\frac{2}{\pi}\left(cot^{-1}\left(0.3\left(cot\left(\frac{\pi(0.05)}{2}\right)\right)+0.3\left(cot\left(\frac{\pi(0.05)}{2}\right)\right)+0.4\left(cot\left(\frac{\pi(0.25)}{2}\right)\right)\right)\right)=0.0738,$

$\frac{2}{\pi}\left(cot^{-1}\left(0.3\left(cot\left(\frac{\pi(0.05)}{2}\right)\right)+0.3\left(cot\left(\frac{\pi(0.05)}{2}\right)\right)+0.4\left(cot\left(\frac{\pi(0.10)}{2}\right)\right)\right)\right)=0.0625,$

$\frac{2}{\pi}\left(cot^{-1}\left(0.3\left(cot\left(\frac{\pi(0.05)}{2}\right)\right)+0.3\left(cot\left(\frac{\pi(0.05)}{2}\right)\right)+0.4\left(cot\left(\frac{\pi(0.15)}{2}\right)\right)\right)\right)=0.0683,$

$\frac{2}{\pi}\left(cot^{-1}\left(0.3\left(cot\left(\frac{\pi(0.05)}{2}\right)\right)+0.3\left(cot\left(\frac{\pi(0.05)}{2}\right)\right)+0.4\left(cot\left(\frac{\pi(0.20)}{2}\right)\right)\right)\right)=0.0716.$

Thus, $T2NNTWAM(\tilde{\mathfrak{Z}}_1,\tilde{\mathfrak{Z}}_2,\tilde{\mathfrak{Z}}_3)=\bigoplus_{\mathfrak{g}=1}^{3}\mathfrak{s}_\mathfrak{g}\tilde{\mathfrak{Z}}_\mathfrak{g}=\langle(0.9244,0.8679,0.9283),(0.1154,0.1252,0.0738),(0.0625,0.0683,0.0716)\rangle.$

2.2.3 T2NN trigonometric weighted geometric mean aggregation operator

In this sub-section, T2NNTWGM is presented.

Definition 4. Let us consider $\tilde{\mathfrak{Z}}_\mathfrak{g}$, which represents a T2NN. Furthermore, we have the associated weight vector $\mathfrak{s}=(\mathfrak{s}_1,\mathfrak{s}_2,\ldots,\mathfrak{s}_\mathfrak{h})$ for $\mathfrak{s}_\mathfrak{g}\ \epsilon\ [0,1]$ and $\sum_{\mathfrak{g}=1}^{\mathfrak{h}}\mathfrak{s}_\mathfrak{g}=1$. Here, the T2NNTWGM operator is defined by Eq. (7):

$$\boldsymbol{T2NNYWGM}(\tilde{\mathfrak{Z}}_1,\tilde{\mathfrak{Z}}_2,\ldots,\tilde{\mathfrak{Z}}_\mathfrak{h})=\otimes_{\mathfrak{g}=1}^{\mathfrak{h}}(\tilde{\mathfrak{Z}}_\mathfrak{g})^{\mathfrak{s}_\mathfrak{g}}. \tag{7}$$

Theorem 4: Let us consider $\tilde{\mathfrak{Z}}_\mathfrak{g}$, denoted as $\tilde{\mathfrak{Z}}_\mathfrak{g}=\langle\left(T_{T_{\tilde{\mathfrak{Z}}_\mathfrak{g}}}(\hbar),T_{I_{\tilde{\mathfrak{Z}}_\mathfrak{g}}}(\hbar),T_{F_{\tilde{\mathfrak{Z}}_\mathfrak{g}}}(\hbar)\right),\left(I_{T_{\tilde{\mathfrak{Z}}_\mathfrak{g}}}(\hbar),I_{I_{\tilde{\mathfrak{Z}}_\mathfrak{g}}}(\hbar),I_{F_{\tilde{\mathfrak{Z}}_\mathfrak{g}}}(\hbar)\right),\left(F_{T_{\tilde{\mathfrak{Z}}_\mathfrak{g}}}(\hbar),F_{I_{\tilde{\mathfrak{Z}}_\mathfrak{g}}}(\hbar),F_{F_{\tilde{\mathfrak{Z}}_\mathfrak{g}}}(\hbar)\right)\rangle$, which represents a T2NN. Furthermore, we have the associated weight vector $\mathfrak{s}=(\mathfrak{s}_1,\mathfrak{s}_2,\ldots,\mathfrak{s}_\mathfrak{h})$ for $\mathfrak{s}_\mathfrak{g}\ \epsilon\ [0,1]$ and $\sum_{\mathfrak{g}=1}^{\mathfrak{h}}\mathfrak{s}_\mathfrak{g}=1$ and $\pi=3.14$. Then, T2NNTWGM operator is described as Eq. (8):

$$\boldsymbol{T2NNTWGM}(\tilde{\mathfrak{Z}}_1,\tilde{\mathfrak{Z}}_2,\ldots,\tilde{\mathfrak{Z}}_\mathfrak{h})=\otimes_{\mathfrak{g}=1}^{\mathfrak{h}}(\tilde{\mathfrak{Z}}_\mathfrak{g})^{\mathfrak{s}_\mathfrak{g}}=$$
$$\left\{\begin{matrix}
\left(\frac{2}{\pi}\left(\boldsymbol{cot^{-1}}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}}\left(\mathfrak{s}_\mathfrak{g}\left(\boldsymbol{cot}\left(\frac{\pi\left(T_{T_{\tilde{\mathfrak{Z}}_\mathfrak{g}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right),\frac{2}{\pi}\left(\boldsymbol{cot^{-1}}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}}\left(\mathfrak{s}_\mathfrak{g}\left(\boldsymbol{cot}\left(\frac{\pi\left(T_{I_{\tilde{\mathfrak{Z}}_\mathfrak{g}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right),\frac{2}{\pi}\left(\boldsymbol{cot^{-1}}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}}\left(\mathfrak{s}_\mathfrak{g}\left(\boldsymbol{cot}\left(\frac{\pi\left(T_{F_{\tilde{\mathfrak{Z}}_\mathfrak{g}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right)\right),\\
\left(\frac{2}{\pi}\left(\boldsymbol{tan^{-1}}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}}\left(\mathfrak{s}_\mathfrak{g}\left(\boldsymbol{tan}\left(\frac{\pi\left(I_{T_{\tilde{\mathfrak{Z}}_\mathfrak{g}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right),\frac{2}{\pi}\left(\boldsymbol{tan^{-1}}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}}\left(\mathfrak{s}_\mathfrak{g}\left(\boldsymbol{tan}\left(\frac{\pi\left(I_{I_{\tilde{\mathfrak{Z}}_\mathfrak{g}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right),\frac{2}{\pi}\left(\boldsymbol{tan^{-1}}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}}\left(\mathfrak{s}_\mathfrak{g}\left(\boldsymbol{tan}\left(\frac{\pi\left(I_{F_{\tilde{\mathfrak{Z}}_\mathfrak{g}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right)\right),\\
\left(\frac{2}{\pi}\left(\boldsymbol{tan^{-1}}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}}\left(\mathfrak{s}_\mathfrak{g}\left(\boldsymbol{tan}\left(\frac{\pi\left(F_{T_{\tilde{\mathfrak{Z}}_\mathfrak{g}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right),\frac{2}{\pi}\left(\boldsymbol{tan^{-1}}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}}\left(\mathfrak{s}_\mathfrak{g}\left(\boldsymbol{tan}\left(\frac{\pi\left(F_{I_{\tilde{\mathfrak{Z}}_\mathfrak{g}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right),\frac{2}{\pi}\left(\boldsymbol{tan^{-1}}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}}\left(\mathfrak{s}_\mathfrak{g}\left(\boldsymbol{tan}\left(\frac{\pi\left(F_{F_{\tilde{\mathfrak{Z}}_\mathfrak{g}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right)\right)
\end{matrix}\right\}. \tag{8}$$

Proof: Proof is shown in Proof-4 in Appendix B.

Theorem 5: The T2NNTWAM operator complies with the following properties:

(i) Monotonicity: Consider that $\tilde{\mathfrak{I}}_{g} = \langle\left(T_{T_{\tilde{\mathfrak{I}}_g}}(\hbar), T_{I_{\tilde{\mathfrak{I}}_g}}(\hbar), T_{F_{\tilde{\mathfrak{I}}_g}}(\hbar)\right), \left(I_{T_{\tilde{\mathfrak{I}}_g}}(\hbar), I_{I_{\tilde{\mathfrak{I}}_g}}(\hbar), I_{F_{\tilde{\mathfrak{I}}_g}}(\hbar)\right), \left(F_{T_{\tilde{\mathfrak{I}}_g}}(\hbar), F_{I_{\tilde{\mathfrak{I}}_g}}(\hbar), F_{F_{\tilde{\mathfrak{I}}_g}}(\hbar)\right)\rangle$ and $\tilde{\mathfrak{I}}_{g}^{*} = \langle\left(T_{T_{\tilde{\mathfrak{I}}_g}^{*}}(\hbar), T_{I_{\tilde{\mathfrak{I}}_g}^{*}}(\hbar), T_{F_{\tilde{\mathfrak{I}}_g}^{*}}(\hbar)\right), \left(I_{T_{\tilde{\mathfrak{I}}_g}^{*}}(\hbar), I_{I_{\tilde{\mathfrak{I}}_g}^{*}}(\hbar), I_{F_{\tilde{\mathfrak{I}}_g}^{*}}(\hbar)\right), \left(F_{T_{\tilde{\mathfrak{I}}_g}^{*}}(\hbar), F_{I_{\tilde{\mathfrak{I}}_g}^{*}}(\hbar), F_{F_{\tilde{\mathfrak{I}}_g}^{*}}(\hbar)\right)\rangle$ are two T2NNs over the universe $\mathfrak{D}$. If $T_{T_{\tilde{\mathfrak{I}}_g}}(\hbar) \geq T_{T_{\tilde{\mathfrak{I}}_g}^{*}}(\hbar), T_{I_{\tilde{\mathfrak{I}}_g}}(\hbar) \geq T_{I_{\tilde{\mathfrak{I}}_g}^{*}}(\hbar), T_{F_{\tilde{\mathfrak{I}}_g}}(\hbar) \geq T_{F_{\tilde{\mathfrak{I}}_g}^{*}}(\hbar)$ and $I_{T_{\tilde{\mathfrak{I}}_g}}(\hbar) \leq I_{T_{\tilde{\mathfrak{I}}_g}^{*}}(\hbar)$, $I_{I_{\tilde{\mathfrak{I}}_g}}(\hbar) \leq I_{I_{\tilde{\mathfrak{I}}_g}^{*}}(\hbar), I_{F_{\tilde{\mathfrak{I}}_g}}(\hbar) \leq I_{F_{\tilde{\mathfrak{I}}_g}^{*}}(\hbar)$ and $F_{T_{\tilde{\mathfrak{I}}_g}}(\hbar) \geq F_{T_{\tilde{\mathfrak{I}}_g}^{*}}(\hbar), F_{I_{\tilde{\mathfrak{I}}_g}}(\hbar) \leq F_{I_{\tilde{\mathfrak{I}}_g}^{*}}(\hbar)$, $F_{F_{\tilde{\mathfrak{I}}_g}}(\hbar) \leq F_{F_{\tilde{\mathfrak{I}}_g}^{*}}(\hbar)$, then $T2NNTWGM(\tilde{\mathfrak{I}}_1, \tilde{\mathfrak{I}}_2, \ldots, \tilde{\mathfrak{I}}_{\mathfrak{h}}) \geq T2NNTWGM(\tilde{\mathfrak{I}}_1^{*}, \tilde{\mathfrak{I}}_2^{*}, \ldots, \tilde{\mathfrak{I}}_{\mathfrak{h}}^{*})$.

Proof: Proof is clear as Proof-2 in Appendix B.

(ii) Idempotency: If $\tilde{\mathfrak{I}}_{g} = \tilde{\mathfrak{I}}$, $T_{T_{\tilde{\mathfrak{I}}}}(\hbar) = T_{T_{\tilde{\mathfrak{I}}_g}}(\hbar), T_{I_{\tilde{\mathfrak{I}}}}(\hbar) = T_{I_{\tilde{\mathfrak{I}}_g}}(\hbar), T_{F_{\tilde{\mathfrak{I}}}}(\hbar) = T_{F_{\tilde{\mathfrak{I}}_g}}(\hbar)$, and $I_{T_{\tilde{\mathfrak{I}}}}(\hbar) = I_{T_{\tilde{\mathfrak{I}}_g}}(\hbar), I_{I_{\tilde{\mathfrak{I}}}}(\hbar) = I_{I_{\tilde{\mathfrak{I}}_g}}(\hbar), I_{F_{\tilde{\mathfrak{I}}}}(\hbar) = I_{F_{\tilde{\mathfrak{I}}_g}}(\hbar)$, and $F_{T_{\tilde{\mathfrak{I}}}}(\hbar) = F_{T_{\tilde{\mathfrak{I}}_g}}(\hbar), F_{I_{\tilde{\mathfrak{I}}}}(\hbar) = F_{I_{\tilde{\mathfrak{I}}_g}}(\hbar)$, $F_{F_{\tilde{\mathfrak{I}}}}(\hbar) = F_{F_{\tilde{\mathfrak{I}}_g}}(\hbar)$, then $T2NNTWGM(\tilde{\mathfrak{I}}_1, \tilde{\mathfrak{I}}_2, \ldots, \tilde{\mathfrak{I}}_{\mathfrak{h}}) = \tilde{\mathfrak{I}}$.

Proof: Proof is clear as Proof-3 in Appendix B.

(iii) Boundedness: Suppose the maximum and minimum T2NNs are defined as:

$\tilde{\mathfrak{I}}_{max} = \langle\left(max_{g}\left(T_{T_{\tilde{\mathfrak{I}}_g}}(\hbar)\right), max_{g}\left(T_{I_{\tilde{\mathfrak{I}}_g}}(\hbar)\right) max_{g}\left(T_{F_{\tilde{\mathfrak{I}}_g}}(\hbar)\right)\right), \left(min_{g}\left(I_{T_{\tilde{\mathfrak{I}}_g}}(\hbar)\right), min_{g}\left(I_{I_{\tilde{\mathfrak{I}}_g}}(\hbar)\right) min_{g}\left(I_{F_{\tilde{\mathfrak{I}}_g}}(\hbar)\right)\right), \left(min_{g}\left(F_{T_{\tilde{\mathfrak{I}}_g}}(\hbar)\right), min_{g}\left(F_{I_{\tilde{\mathfrak{I}}_g}}(\hbar)\right) min_{g}\left(F_{F_{\tilde{\mathfrak{I}}_g}}(\hbar)\right)\right)\rangle$, $\tilde{\mathfrak{I}}_{min} = \langle\left(min_{g}\left(T_{T_{\tilde{\mathfrak{I}}_g}}(\hbar)\right), min_{g}\left(T_{I_{\tilde{\mathfrak{I}}_g}}(\hbar)\right) min_{g}\left(T_{F_{\tilde{\mathfrak{I}}_g}}(\hbar)\right)\right), \left(max_{g}\left(I_{T_{\tilde{\mathfrak{I}}_g}}(\hbar)\right), max_{g}\left(I_{I_{\tilde{\mathfrak{I}}_g}}(\hbar)\right) max_{g}\left(I_{F_{\tilde{\mathfrak{I}}_g}}(\hbar)\right)\right), \left(max_{g}\left(F_{T_{\tilde{\mathfrak{I}}_g}}(\hbar)\right), max_{g}\left(F_{I_{\tilde{\mathfrak{I}}_g}}(\hbar)\right) max_{g}\left(F_{F_{\tilde{\mathfrak{I}}_g}}(\hbar)\right)\right)\rangle$, then , $\tilde{\mathfrak{I}}_{max} \leq \mathrm{T2}NNTWGM(\tilde{\mathfrak{I}}_1, \tilde{\mathfrak{I}}_2, \ldots, \tilde{\mathfrak{I}}_{\mathfrak{h}}) \leq \tilde{\mathfrak{I}}_{min}$ can hold.

Proof: Proof is clear.

Numerical example: The weights of the three decision-makers $(\mathfrak{s}_{g})$ are 0.3, 0.3, and 0.4, respectively. The T2NN assigned by the decision-makers are $\tilde{\mathfrak{I}}_1 = \langle(0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05)\rangle$, $\tilde{\mathfrak{I}}_1 = \langle(0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05)\rangle$ and $\tilde{\mathfrak{I}}_3 = \langle(0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20)\rangle$. The procedures of the T2NNTWGM aggregation operator are as follows:

$$\frac{2}{\pi}\left(cot^{-1}\left(0.3\left(cot\left(\frac{\pi(0.95)}{2}\right)\right) + 0.3\left(cot\left(\frac{\pi(0.95)}{2}\right)\right) + 0.4\left(cot\left(\frac{\pi(0.70)}{2}\right)\right)\right)\right) = 0.8434,$$

$$\frac{2}{\pi}\left(cot^{-1}\left(0.3\left(cot\left(\frac{\pi(0.90)}{2}\right)\right) + 0.3\left(cot\left(\frac{\pi(0.90)}{2}\right)\right) + 0.4\left(cot\left(\frac{\pi(0.75)}{2}\right)\right)\right)\right) = 0.8376,$$

$$\frac{2}{\pi}\left(cot^{-1}\left(0.3\left(cot\left(\frac{\pi(0.95)}{2}\right)\right) + 0.3\left(cot\left(\frac{\pi(0.95)}{2}\right)\right) + 0.4\left(cot\left(\frac{\pi(0.80)}{2}\right)\right)\right)\right) = 0.8884,$$

$$\frac{2}{\pi}\left(tan^{-1}\left(0.3\left(tan\left(\frac{\pi(0.10)}{2}\right)\right) + 0.3\left(tan\left(\frac{\pi(0.10)}{2}\right)\right) + 0.4\left(tan\left(\frac{\pi(0.15)}{2}\right)\right)\right)\right) = 0.1202,$$

$$\frac{2}{\pi}\left(tan^{-1}\left(0.3\left(tan\left(\frac{\pi(0.10)}{2}\right)\right) + 0.3\left(tan\left(\frac{\pi(0.10)}{2}\right)\right) + 0.4\left(tan\left(\frac{\pi(0.20)}{2}\right)\right)\right)\right) = 0.1409,$$

$$\frac{2}{\pi}\left(tan^{-1}\left(0.3\left(tan\left(\frac{\pi(0.05)}{2}\right)\right) + 0.3\left(tan\left(\frac{\pi(0.05)}{2}\right)\right) + 0.4\left(tan\left(\frac{\pi(0.25)}{2}\right)\right)\right)\right) = 0.1335,$$

$\frac{2}{\pi}\left(tan^{-1}\left(0.3\left(tan\left(\frac{\pi(0.05)}{2}\right)\right)+0.3\left(tan\left(\frac{\pi(0.05)}{2}\right)\right)+0.4\left(tan\left(\frac{\pi(0.10)}{2}\right)\right)\right)\right)=0.0701,$
$\frac{2}{\pi}\left(tan^{-1}\left(0.3\left(tan\left(\frac{\pi(0.05)}{2}\right)\right)+0.3\left(tan\left(\frac{\pi(0.05)}{2}\right)\right)+0.4\left(tan\left(\frac{\pi(0.15)}{2}\right)\right)\right)\right)=0.0906,$
$\frac{2}{\pi}\left(tan^{-1}\left(0.3\left(tan\left(\frac{\pi(0.05)}{2}\right)\right)+0.3\left(tan\left(\frac{\pi(0.05)}{2}\right)\right)+0.4\left(tan\left(\frac{\pi(0.20)}{2}\right)\right)\right)\right)=0.1116.$

Thus, $T2NNTWGM(\tilde{\mathrm{a}}_1,\tilde{\mathrm{a}}_2,\tilde{\mathrm{a}}_3)=\oplus_{g=1}^{3}\mathfrak{s}_g\tilde{\mathrm{a}}_g=\langle(0.8434,0.8376,0.8884),(0.1202,0.1409,0.1335),(0.0701,0.0906,0.1116)\rangle$.

2.3 Novel T2NN-LOPCOW-CoCoSo Hybrid Method

The T2NN-LOPCOW-CoCoSo method has been enhanced for the assessment of sustainable energy production performance in HPPs. Let $(\mathcal{H}_x)=\{\mathcal{H}_1,\mathcal{H}_2,\ldots,\mathcal{H}_m\}\ (x=1,2,\ldots,m)$ be alternatives and $\left(\mathcal{C}_{y^{(ql)}}\right)=\left\{\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_{n^{(ql)}}\right\}\ (y^{(ql)}=1,2,\ldots,n^{(ql)})$ be qualitative criteria and $\left(\mathcal{C}_{y^{(qn)}}\right)=\left\{\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_{n^{(qn)}}\right\}\ (y^{(qn)}=1,2,\ldots,n^{(qn)})$ be quantitative criteria and $(\mathcal{C}_y)=\{\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n\}\ (y=1,2,\ldots,n)$ be total criteria and $(\mathcal{E}_z)=\{\mathcal{E}_1,\mathcal{E}_2,\ldots,\mathcal{E}_\sigma\}\ (z=1,2,\ldots,\sigma)$ for the assessment of sustainable energy production performance in HPPs. For application of the T2NN-LOPCOW-CoCoSo method, the T2NNTWAM and T2NNTWGM operators are employed to aggregate matrices based on subjective expert evaluations. These operators, which are based on trigonometric operations, enable more precise computations during the aggregation process, resulting in stronger and more reliable aggregated matrices. The steps of the trigonometric operation-based-T2NN-LOPCOW-CoCoSo hybrid method are given in the following part.

<u>*Establishing the relative weights of the experts*</u>:

Step 1: The assessment of experts' experience and expertise is based on linguistic variables (LVs) as detailed in Table 1. These are subsequently transformed to T2NNs, and significance weights are assigned to each expert within the T2NN framework.

Table 1
Linguistic variables denoting levels of experience and expertise [30]

Experience (years)	Expertise	T2NNs
5<	Very-poor (VP)	⟨(0.2,0.2,0.1), (0.65,0.8,0.85), (0.45,0.8,0.7)⟩
[5,10)	Poor (P)	⟨(0.35,0.35,0.1), (0.5,0.75,0.8), (0.5,0.75,0.65)⟩
[10,15)	Medium-poor (MP)	⟨(0.4,0.3,0.35), (0.5,0.45,0.6), (0.45,0.4,0.6)⟩
[15,20)	Medium (M)	⟨(0.5,0.45,0.5), (0.4,0.35,0.5), (0.35,0.3,0.45)⟩
[20,25)	Medium-good (MG)	⟨(0.6,0.45,0.5), (0.2,0.15,0.25), (0.1,0.25,0.15)⟩
[25,30)	Good (G)	⟨(0.7,0.75,0.8), (0.15,0.2,0.25), (0.1,0.15,0.2)⟩
≥30	Very-good (VG)	⟨(0.95,0.9,0.95), (0.1,0.1,0.05), (0.05,0.05,0.05)⟩

Step 2: The aggregation operator is employed to consolidate the experts qualifications: $T2NNTWAM(\tilde{\mathrm{a}}_1,\ldots,\tilde{\mathrm{a}}_\sigma)=\oplus_{z=1}^{\sigma=2}\mathfrak{s}_z\tilde{\mathrm{a}}_z$, which is delineated in Eq. (6). Here, it is supposed that the levels of expertise and experience among the experts bear equal significance. Therefore, $\mathfrak{s}_1=\mathfrak{s}_2=0.5$, $\mathfrak{s}_1,\mathfrak{s}_2\in[0,1]$, and $\mathfrak{s}_1+\mathfrak{s}_2=1$ and the parameter is $\pi=3.14$.

Step 3: The score $\mathcal{S}(\tilde{\mathrm{a}}_z)$ and accuracy $\mathcal{A}(\tilde{\mathrm{a}}_z)$ functions are computed.

Step 4: The corresponding matrix $(\mathbb{E}=[\mathbb{E}_z]_\sigma)$ of the experts is calculated by performing Eq. (9):

$$\mathbb{E}_z=\frac{\mathcal{S}(\tilde{\mathrm{a}}_z)}{\sum_{z=1}^{\sigma}\mathcal{S}(\tilde{\mathrm{a}}_z)}\ ;z=(1,2,\ldots,\sigma). \tag{9}$$

Determination of criterion weights using the trigonometric operation-based T2NN-LOPCOW method:

Step 5: Alternatives $(\mathcal{H}_x)$ are assessed by every expert $(\mathcal{E}_z)$ regarding each qualitative criteria $\left(\mathcal{C}_{\mathcal{y}^{(ql)}}\right)$, utilizing Table 2. Following this, initial matrices $\left(\tilde{\mathcal{F}}^{(z)} = \left[\tilde{\mathcal{F}}^{(z)}{}_{x\mathcal{y}^{(ql)}}\right]_{mxn^{(ql)}}\right)$ are constructed for the qualitative criteria. Here, $\tilde{\mathcal{F}}^{(z)}{}_{x\mathcal{y}^{(ql)}} = \langle \left(T_{T_{\tilde{\mathcal{F}}^{(z)}{}_{x\mathcal{y}^{(ql)}}}}(\hbar), T_{I_{\tilde{\mathcal{F}}^{(z)}{}_{x\mathcal{y}^{(ql)}}}}(\hbar), T_{F_{\tilde{\mathcal{F}}^{(z)}{}_{x\mathcal{y}^{(ql)}}}}(\hbar)\right), \left(I_{T_{\tilde{\mathcal{F}}^{(z)}{}_{x\mathcal{y}^{(ql)}}}}(\hbar), I_{I_{\tilde{\mathcal{F}}^{(z)}{}_{x\mathcal{y}^{(ql)}}}}(\hbar), I_{F_{\tilde{\mathcal{F}}^{(z)}{}_{x\mathcal{y}^{(ql)}}}}(\hbar)\right), \left(F_{T_{\tilde{\mathcal{F}}^{(z)}{}_{x\mathcal{y}^{(ql)}}}}(\hbar), F_{I_{\tilde{\mathcal{F}}^{(z)}{}_{x\mathcal{y}^{(ql)}}}}(\hbar), F_{F_{\tilde{\mathcal{F}}^{(z)}{}_{x\mathcal{y}^{(ql)}}}}(\hbar)\right)\rangle$,where $\left(x = 1, \ldots, m;\ \mathcal{y}^{(ql)} = 1, \ldots, n^{(ql)};\ z = 1, \ldots, \sigma\right)$.

Table 2
Linguistic variables for assessing alternatives based on qualitative criteria [30]

Linguistic variables	T2NNs
Very low (VL)	(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)
Low (L)	(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)
Medium low (ML)	(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)
Medium (M)	(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)
Medium high (MH)	(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)
High (H)	(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)
Very high (VH)	(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)

Step 6: Using the T2NNTWAM operator showed in Eq. (6), detemine the aggregated matrix $\left(\tilde{\mathcal{G}} = \left[\tilde{\mathcal{G}}_{x\mathcal{y}^{(ql)}}\right]_{mxn^{(ql)}}\right)$.

Step 7: The score $\mathcal{S}(\tilde{\mathcal{G}})$ accuracy $\mathcal{A}(\tilde{\mathcal{G}})$ functions are calculated. Then, create the numeric matrix $\left(\mathcal{S}(\tilde{\mathcal{G}}) = \left[\mathcal{S}(\tilde{\mathcal{G}})_{x\mathcal{y}^{(ql)}}\right]_{mxn^{(ql)}}\right)$.

Step 8: Using the numeric matrix for qualitative criteria $\left(\mathcal{S}(\tilde{\mathcal{G}}) = \left[\mathcal{S}(\tilde{\mathcal{G}})_{x\mathcal{y}^{(ql)}}\right]_{mxn^{(ql)}}\right)$ and the initial matrix for quantitative criteria $\left(\mathcal{G} = \left[\mathcal{G}_{x\mathcal{y}^{(qn)}}\right]_{mxn^{(qn)}}\right)$, obtain the overall initial matrix $\left(\mathcal{H} = \left[\mathcal{H}_{x\mathcal{y}}\right]_{mxn}\right)$.

Step 9: Using the overall initial matrix, the normalized decision matrix $\left(\mathcal{H}^{(N)} = \left[\mathcal{H}^{(N)}{}_{x\mathcal{y}}\right]_{mxn}\right)$ is performed by using Eq. (10), Eq. (11) and Eq. (12) $(0 \leq \varrho \leq 1)$:

$$\boldsymbol{\mathcal{H}^{(N)}{}_{x\mathcal{y}} = \varrho\left(\mathcal{H}^{(1)}{}_{x\mathcal{y}}\right) + (1 - \varrho)\left(\mathcal{H}^{(2)}{}_{x\mathcal{y}}\right); (x = 1, 2, \ldots, m;\ \mathcal{y} = 1, 2, \ldots, n)}, \tag{10}$$

$$\boldsymbol{\mathcal{H}^{(1)}{}_{x\mathcal{y}} = \begin{cases} \dfrac{max_{1\leq t\leq m}(\mathcal{H}_{t\mathcal{y}}) - (\mathcal{H}_{x\mathcal{y}})}{max_{1\leq t\leq m}(\mathcal{H}_{t\mathcal{y}}) - min_{1\leq t\leq m}(\mathcal{H}_{t\mathcal{y}})}, \mathcal{C}_{\mathcal{y}} \in \mathcal{C}^{-} \\ \dfrac{(\mathcal{H}_{x\mathcal{y}}) - min_{1\leq t\leq m}(\mathcal{H}_{t\mathcal{y}})}{max_{1\leq t\leq m}(\mathcal{H}_{t\mathcal{y}}) - min_{1\leq t\leq m}(\mathcal{H}_{t\mathcal{y}})}, \mathcal{C}_{\mathcal{y}} \in \mathcal{C}^{+} \end{cases}; (x = 1, 2, \ldots, m;\ \mathcal{y} = 1, 2, \ldots, n)}, \tag{11}$$

$$\boldsymbol{\mathcal{H}^{(2)}{}_{x\mathcal{y}} = \begin{cases} \dfrac{\frac{1}{(\mathcal{H}_{x\mathcal{y}})}}{\sum_{t=1}^{m}\frac{1}{(\mathcal{H}_{t\mathcal{y}})}}, \mathcal{C}_{\mathcal{y}} \in \mathcal{C}^{-} \\ \dfrac{(\mathcal{H}_{x\mathcal{y}})}{\sum_{t=1}^{m}(\mathcal{H}_{t\mathcal{y}})}, \mathcal{C}_{\mathcal{y}} \in \mathcal{C}^{+} \end{cases}; (x = 1, 2, \ldots, m;\ \mathcal{y} = 1, 2, \ldots, n)}. \tag{12}$$

Step 10: Using the normalized matrix, the root means square value matrix $\left(\mathcal{R} = \left[\mathcal{R}_{y}\right]_{n}\right)$ and the standard deviation matrix $\left(\mathcal{O} = \left[\mathcal{O}_{y}\right]_{n}\right)$ are calculated with Eq. (13) and Eq. (14), respectively:

$$\boldsymbol{\mathcal{R}_{y}} = \sqrt{\frac{1}{m}\sum_{x=1}^{m}\left(\mathcal{H}^{(N)}{}_{xy}\right)^{2}}\,;\,(y = 1,2,\dots,n), \tag{13}$$

$$\boldsymbol{\mathcal{O}_{y}} = \sqrt{\frac{1}{m-1}\sum_{x=1}^{m}\left(\left(\mathcal{H}^{(N)}{}_{xy}\right) - \frac{1}{m}\sum_{t=1}^{m}\left(\mathcal{H}^{(N)}{}_{ty}\right)\right)^{2}}\,;\,(y = 1,2,\dots,n). \tag{14}$$

Then, the contribution percentage of the criteria $\left(\mathcal{U} = \left[\mathcal{U}_{y}\right]_{n}\right)$ is computed by performing the Eq (18):

$$\boldsymbol{\mathcal{U}_{y}} = \left|\ln\left(\frac{\mathcal{R}_{y}}{\mathcal{O}_{y}}\right) X\ 100\%\right|;\ (y = 1,2,\dots,n). \tag{15}$$

Step 11: Using the contribution percentage of the criteria matrix $\left(\mathcal{U} = \left[\mathcal{U}_{y}\right]_{n}\right)$, the weighting matrix $\left(w = \left[w_{y}\right]_{n}\right)$ is calculated with Eq. (16):

$$\boldsymbol{T2NNYWGM}\left(\tilde{\mathfrak{I}}_{1}, \tilde{\mathfrak{I}}_{2}, \dots, \tilde{\mathfrak{I}}_{\mathfrak{h}}\right) = \otimes_{\mathfrak{g}=1}^{\mathfrak{h}}\left(\tilde{\mathfrak{I}}_{\mathfrak{g}}\right)^{\mathfrak{s}_{\mathfrak{g}}}. \tag{16}$$

where $\sum_{y=1}^{n} w_{y} = 1$ and $w_{y} \in [0,1]$ $(y = 1,2,\dots,n)$.

<u>*Ranking of Alternatives via the Trigonometric Operation-Based T2NN-CoCoSo Method*</u>:

Step 12: Using the overall matrix for all criteria $\left(\mathcal{H} = \left[\mathcal{H}_{xy}\right]_{mxn}\right)$, the normalized matrix $\left(\mathcal{K} = \left[\mathcal{K}_{xy}\right]_{mxn}\right)$ is calculated with Eq. (17):

$$\boldsymbol{\mathcal{K}_{xy}} = \begin{cases} \frac{\mathcal{H}_{xy} - min_{x}(\mathcal{H}_{xy})}{max_{x}(\mathcal{H}_{xy}) - min_{x}(\mathcal{H}_{xy})}, \boldsymbol{if\ y \in benefit} \\ \frac{max_{x}(\mathcal{H}_{xy}) - \mathcal{H}_{xy}}{max_{x}(\mathcal{H}_{xy}) - min_{x}(\mathcal{H}_{xy})}\ \boldsymbol{if\ y \in cost} \end{cases};\,(x = 1,2,\dots,m;\ y = 1,2,\dots,n). \tag{17}$$

Step 13: The cumulative value of the weighted comparability sequence matrix $(\mathbb{O} = [\mathbb{O}_{x}]_{m})$ is calculated using Eq. (18):

$$\mathbb{O}_{x} = \sum_{y=1}^{n} w_{y}\left(\mathcal{K}_{xy}\right);\,(x = 1,2,\dots,m;\ y = 1,2,\dots,n). \tag{18}$$

Step 14: The combined influence of the power weighting in the comparability sequence matrix $(\mathbb{P} = [\mathbb{P}_{x}]_{m})$ is calculated using Eq. (19):

$$\mathbb{P}_{x} = \sum_{y=1}^{n} w_{y}^{\left(\mathcal{K}_{xy}\right)};\,(x = 1,2,\dots,m;\ y = 1,2,\dots,n). \tag{19}$$

Step 15: The relative importance of the alternatives through the utilization of scoring strategies for each alternative is calculated using Eq. (20), Eq. (21), and Eq. (22):

$$\mathbb{R}_{xa} = \frac{\mathbb{O}_x+\mathbb{P}_x}{\sum_{x=1}^{m}(\mathbb{O}_x+\mathbb{P}_x)}; (\boldsymbol{x = 1, 2, \ldots, m}), \tag{7}$$

$$\mathbb{R}_{xb} = \frac{\mathbb{O}_x}{\boldsymbol{min}_x(\mathbb{O}_x)} + \frac{\mathbb{P}_x}{\boldsymbol{min}_x(\mathbb{P}_x)}; (\boldsymbol{x = 1, 2, \ldots, m}), \tag{7}$$

$$\mathbb{R}_{xc} = \frac{\boldsymbol{\psi}(\mathbb{O}_x)+(\boldsymbol{1-\psi})\mathbb{P}_x}{\boldsymbol{\psi}\left(\boldsymbol{max}_x(\mathbb{O}_x)\right)+\left((\boldsymbol{1-\psi})\left(\boldsymbol{max}_x(\mathbb{P}_x)\right)\right)}; (\boldsymbol{x = 1, 2, \ldots, m};\ \boldsymbol{0 \le \psi \le 1}), \tag{7}$$

Step 16: The overall score value matrix ($\mathbb{R} = [\mathbb{R}_x]_m$) is calculated using Eq. (23):

$$\mathbb{R}_x = \left(\sqrt[3]{(\mathbb{R}_{xa})(\mathbb{R}_{xb})(\mathbb{R}_{xc})}\right) + \left(\frac{(\mathbb{R}_{xa})+(\mathbb{R}_{xb})+(\mathbb{R}_{xc})}{\boldsymbol{3}}\right);\ (\boldsymbol{x = 1, 2, \ldots, m}), \tag{7}$$

and then the highest value of $\mathbb{R}_x$ is the best alternative.

2.4 Sensitivity Analysis for Robustness

At this phase, a robustness test of the trigonometric operation-based-T2NN-LOPCOW-CoCoSo hybrid method developed for evaluating the sustainability performance of HPPs is conducted. Within this framework, sensitivity analysis scenarios are designed to assess how changes in conditions affect the sustainability performance of HPPs. Two scenarios are planned. The first scenario is based on variations in the ϱ parameter, which plays a significant role in determining the importance levels of criteria. The second scenario is designed to identify changes in the ψ parameter, which plays a crucial role in determining the ranking of alternatives. This way, the robustness levels of the developed hybrid model are determined.

3. Case Study

To demonstrate and test T2NN-LOPCOW-CoCoSo hybrid model based on trigonometric operations, developed to assess the sustainable performance of HPPs, a case study was designed. This case study specifically targeted the HPPs located within the province of Artvin, Türkiye. Due to the geographical structure of Artvin, it hosts numerous rivers, and multiple HPPs were established along these rivers for sustainable energy production. This case study was developed to address not only the technical performance but also the sustainable performance of these HPPs. Within this context, 12 criteria were defined, consisting of three quantitative and nine qualitative criteria. The sustainable performance of 17 HPPs falling within the 10 MW to 50 MW installed capacity range was assessed. Additionally, a local administrators and engineers was formed to evaluate the qualitative criteria. During this phase, experts were provided with in-depth information about the criteria and the HPPs under assessment.

3.1 Decision Model for the Artvin Province

3.1.1 Experts

The aim was to establish a hybrid expert group for the assessment of the sustainable performance levels of HPPs in the range of 10 MW to 50 MW installed capacity within Türkiye's Artvin province. This expert group was composed of various professionals to ensure a comprehensive evaluation. Local administrators represented the expectations and viewpoints of the local community. Additionally, project managers and engineers involved in the installation and operation of HPPs were included in the expert group. In the context of environmental impacts, an environmental engineer was also part of the expert team. The expert group consisted of 10 members. Table 3 provides an overview of the expertise, experience, and professions of the expert group members.

Table 3
Expert panel

Experts	Experience (Year)	Expertise	Professions
$\mathcal{E}_1$	11	MG	Local administrator
$\mathcal{E}_2$	14	MG	Energy systems engineer
$\mathcal{E}_3$	18	G	Local administrator
$\mathcal{E}_4$	21	G	Environmental engineer
$\mathcal{E}_5$	25	VG	Electrical engineer
$\mathcal{E}_6$	24	VG	Project Manager
$\mathcal{E}_7$	20	G	Local administrator
$\mathcal{E}_8$	17	MG	Construction engineer
$\mathcal{E}_9$	22	G	Local administrator
$\mathcal{E}_{10}$	16	MG	Local administrator

The selection process for the expert group involved discussions with politicians and policymakers in the Artvin province, which led to the identification of the expert group members. Through consultations with the expert group, each HPP was evaluated based on qualitative criteria, and decision matrices were established. Quantitative criteria were also determined and incorporated into the decision matrices. Once the final decision matrices were defined, input decision matrices were created, and the desired outputs were identified.

3.1.2 Criterion identification

Two approaches were adopted in determining the criteria for assessing the sustainable performance of HPPs. The first approach focused on the technical aspects, primarily the energy production outputs of HPPs. In this approach, the criteria were established as quantitative criteria and defined as such. The second approach, on the other hand, considered HPPs from an environmental and social perspective. In this approach, the criteria were described as qualitative criteria. The first three criteria are quantitative and entirely benefit-based. The remaining nine criteria consist of five cost-based criteria and four benefit-based criteria. Consequently, a total of 12 criteria were formulated for the evaluation of the sustainable performance of HPPs. These criteria are elucidated as follows:

i. *Installed capacity* $\left(\mathcal{C}_{1(qn)}\right)$ – HPPs produce electricity through turbines and generators, with their installed capacity (expressed in MW or GW) indicating the maximum potential output, which, along with environmental sustainability, depends on factors such as water source and turbine technology.
ii. *The annual production capacity* $\left(\mathcal{C}_{2(qn)}\right)$ – The annual production capacity of a HPP, typically expressed in MW, reflects its yearly electricity generation potential, primarily determined by installed capacity, water source size, flow rate, discharge, and seasonal variations, and is a key parameter in performance assessment.
iii. *The average annual production* $\left(\mathcal{C}_{3(qn)}\right)$ – The average annual production of a HPP represents the mean electricity generated over past years, influenced by installed and annual production capacities, maintenance schedules, water flow variations, electricity demand, and environmental and social factors, and is used to assess operational efficiency.
iv. *Adverse impact on fish populations and behaviors* $\left(\mathcal{C}_{4(ql)}\right)$ – This criterion evaluates the impacts of a HPP on fish passages, migration behaviors, populations, and habitats, and is classified among cost-based and qualitative criteria due to its focus on negative ecological effects in the plant's region.

v. *Adverse impact on cultural and historical heritage* $\left(\mathcal{C}_{5(ql)}\right)$ – This criterion assesses the adverse impacts of a HPP on the cultural and historical heritage of its region, including potential damage to archaeological sites, reduced tourism appeal, and challenges in preserving heritage, and is therefore classified among cost-based and qualitative criteria in sustainability evaluations.

vi. *Disaster risk* $\left(\mathcal{C}_{6(ql)}\right)$ – This criterion evaluates the potential disaster risks of a HPP, including floods, water overflow, landslides, and other calamities arising from dam construction or inadequate planning, and is considered an adverse factor in assessing the HPP's sustainable performance.

vii. *Adverse impact on the use of water resources* $\left(\mathcal{C}_{7(ql)}\right)$ – This criterion addresses the adverse impacts of HPPs on regional water resources, including disruptions to flow regulation, ecosystem integrity, water quality, and the sustainable use of water for future generations, and is thus classified as both a qualitative and cost-based criterion.

viii. *Adverse impact on local community health and safety* $\left(\mathcal{C}_{8(ql)}\right)$ – This criterion evaluates the adverse impacts of HPPs on the health and safety of local communities, including noise, vibrations, water contamination, and risks of erosion or landslides, and is therefore classified as a qualitative and cost-based criterion in assessing HPPs' sustainable performance.

ix. *Local community involvement* $\left(\mathcal{C}_{9(ql)}\right)$ – This criterion addresses the need for collaboration and effective communication with the local community during HPP planning and implementation, emphasizing transparent participation processes, consideration of community concerns, and the provision of social benefits, and is thus classified as a qualitative criterion in assessing sustainable performance.

x. *Contribution to the local economy* $\left(\mathcal{C}_{10(ql)}\right)$ – This criterion evaluates the economic impacts of HPP projects on local communities, including tax benefits, access to low-cost electricity, employment opportunities, support for local suppliers, and contributions to regional development, and is therefore classified as a qualitative criterion in assessing HPPs' sustainable performance.

xi. *Contribution to transportation infrastructure* $\left(\mathcal{C}_{11(ql)}\right)$ – This criterion evaluates the impacts of HPP projects on regional transportation infrastructure, accessibility, and community daily life, considering both the creation of new roads for HPP operations and their benefits to the region, and is therefore classified as a benefit-based and qualitative criterion.

xii. *Waste management* $\left(\mathcal{C}_{12(ql)}\right)$ – This criterion assesses the success of HPPs in managing all operational waste, including generation, storage, transport, disposal, chemical waste minimization, wastewater treatment, and recycling efforts, and is therefore classified as a benefit-based and qualitative criterion.

3.1.3 Hydroelectric power plants in Artvin

While the primary focus of this research lies in the evaluation of the sustainable performance of HPPs, in order to enhance comparability among HPPs, a specific range of HPPs with an installed capacity ranging from 10 MW to 50 MW has been selected. Within the province of Artvin, a total of seventeen HPPs fall within this category. These HPPs are situated in various locations throughout the Artvin province, spanning different rivers. The identified HPPs are as follows:

i. *Erenler HPP* $(\mathcal{H}_1)$ – The Erenler HPP in central Artvin, Türkiye, operated by BME United Contractors, has a 45 MW capacity and an average annual production of 96,225,815 kWh, sufficient to supply electricity for approximately 26,500 people or 32,200 households [31].
ii. *Arpa HPP* $(\mathcal{H}_2)$ – The Arpa HPP on the Arpa Stream in Borçka, Artvin, Türkiye, operated by Pokut Elektrik, has a 32.41 MW capacity and an average annual production of 43,132,157 kWh, supplying electricity for approximately 11,900 people or 14,400 households [31].
iii. *Artvin Kalecik HPP* $(\mathcal{H}_3)$ – The Artvin Kalecik HPP on the Karçal Stream in Şavşat, Artvin, Türkiye, operated by Dağlar Enerji, has a 27.50 MW capacity and an average annual production of 71,150,000 kWh, supplying electricity for approximately 19,600 people or 23,800 households [31].
iv. *Papart HPP* $(\mathcal{H}_4)$ – The Papart HPP on the Papart Stream in Şavşat, Artvin, Türkiye, owned by Elite Elektrik, has a 26.60 MW capacity and an average annual production of 68,042,818 kWh, supplying electricity for approximately 18,700 people or 22,800 households [31].
v. *Erenköy HPP* $(\mathcal{H}_5)$ – The Erenköy HPP on the Kabaca Stream in Murgul, Artvin, Türkiye, operated by RHG Enertürk, has a 21.46 MW capacity (planned to reach 23 MW at full operation) and an average annual production of 63,807,641 kWh, supplying electricity for approximately 17,600 people or 21,400 households [31].
vi. *Murgul HPP* $(\mathcal{H}_6)$ – The Murgul HPP on the Kabaca Stream in Murgul, Artvin, Türkiye, owned by Cengiz Enerji, has a 19.60 MW capacity and an average annual production of 54,679,662 kWh, supplying electricity for approximately 15,100 people or 18,300 households [31].
vii. *Diyoban HPP* $(\mathcal{H}_7)$ – The Diyoban HPP on the Papart Stream in Şavşat, Artvin, Türkiye, owned by Ati İnşaat Enerji, has a 19.04 MW capacity and an average annual production of 33,887,969 kWh, supplying electricity for approximately 9,330 people or 11,345 households [31].
viii. *Çakırlar HPP* $(\mathcal{H}_8)$ – The Çakırlar HPP on the Kabaca Stream in Murgul, Artvin, Türkiye, owned by Gama Enerji, has a 16.21 MW capacity and an average annual production of 48,132,885 kWh, supplying electricity for approximately 13,250 people or 16,100 households [31].
ix. *Cala HPP* $(\mathcal{H}_9)$ – The Cala HPP on the Yüncüler (Hüngamek) Stream in Yusufeli, Artvin, Türkiye, has an average annual production of 21,671,498 kWh, supplying electricity for approximately 5,970 people or 7,255 households [31].
x. *Orta HPP* $(\mathcal{H}_{10})$ – The Orta HPP in Arhavi, Artvin, Türkiye, has an average annual production of 40,918,180 kWh, supplying electricity for approximately 11,300 people or 13,700 households [31].
xi. *Erik HPP* $(\mathcal{H}_{11})$ – The Erik HPP on the Meydancık River in Şavşat, Artvin, Türkiye, has an average annual production of 41,273,000 kWh, supplying electricity for approximately 11,360 people or 13,820 households [31].
xii. *Şavşat HPP* $(\mathcal{H}_{12})$ – The Şavşat HPP on the Şavşat River in Artvin, Türkiye, owned by 2M Energy, has a 14.52 MW capacity (planned to reach 16 MW) and an average annual production of 44,512,240 kWh, supplying electricity for approximately 12,260 people or 14,900 households [31].
xiii. *Taşköprü HPP* $(\mathcal{H}_{13})$ – The Taşköprü HPP on the Aralık Creek in Borçka, Artvin, Türkiye, has an average annual production of 24,227,537 kWh, supplying electricity for approximately 6,670 people or 8,110 households [31].

xiv. *Aralık HPP* $(\mathcal{H}_{14})$ – The Aralık HPP on the Aralık Creek in Borçka, Artvin, Türkiye, owned by Energo Pro, has a 12.41 MW capacity and an average annual production of 47,826,776 kWh, supplying electricity for approximately 13,170 people or 16,010 households [31].

xv. *Cansu HPP* $(\mathcal{H}_{15})$ – The Cansu HPP on the Kabaca Creek in Murgul, Artvin, Türkiye, owned by Cansu Elektrik, has an 11 MW capacity and an average annual production of 39,558,132 kWh, supplying electricity for approximately 10,890 people or 13,240 households [31].

xvi. *İskale HPP* $(\mathcal{H}_{16})$ – The İskale HPP in Murgul, Artvin, Türkiye, has an average annual production of 25,223,790 kWh, supplying electricity for approximately 6,945 people or 8,445 households [31].

xvii. *Kavak HPP* $(\mathcal{H}_{17})$ – The Kavak HPP on the Orçi Creek in Arhavi, Artvin, Türkiye, has an average annual production of 22,284,275 kWh, supplying electricity for approximately 6,140 people or 7,460 households [31].

Fig. 2 presents the criteria and alternatives incorporated in the decision model for evaluating the sustainability performance of HPPs in Artvin province, Türkiye.

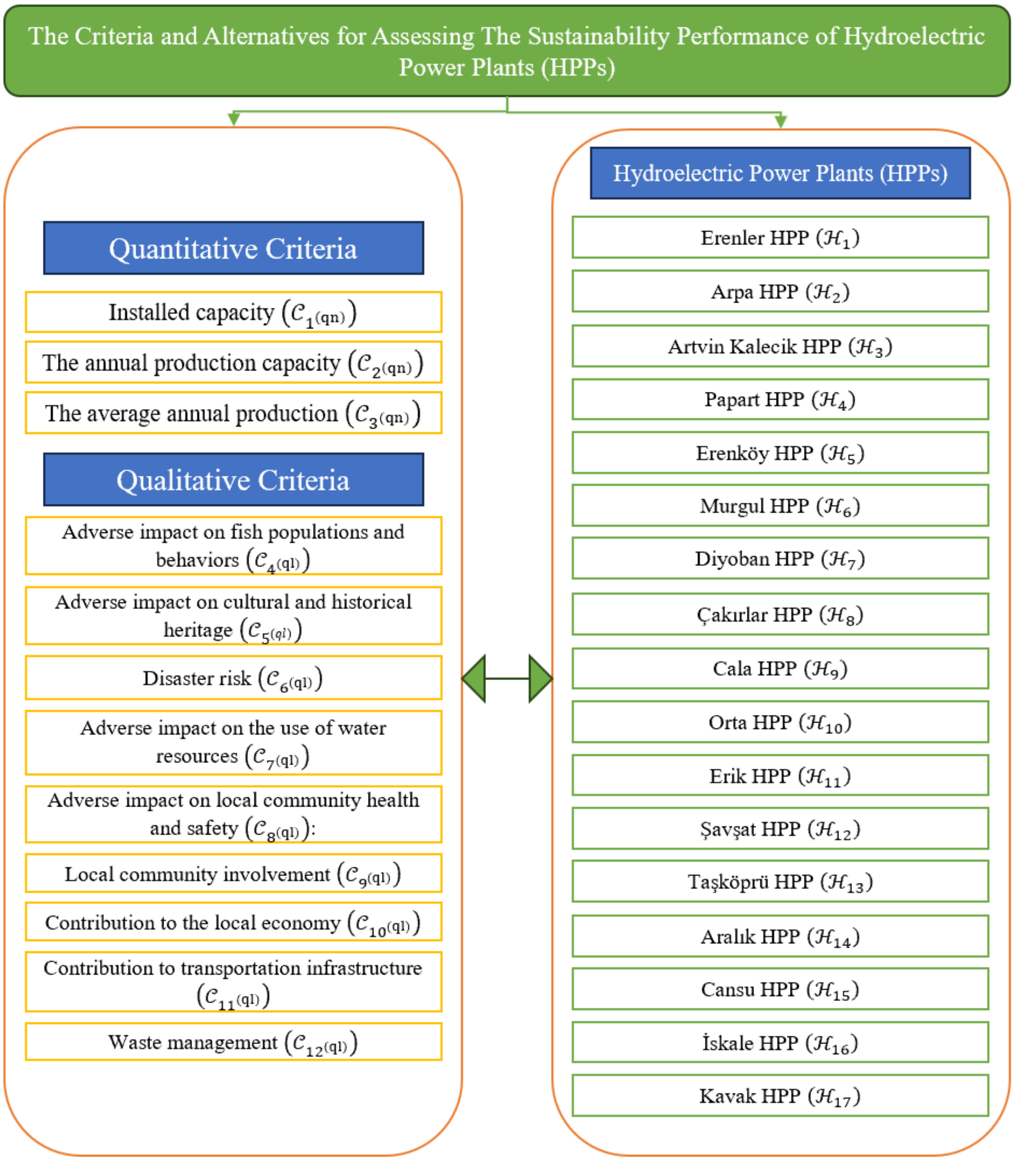

Fig. 2. The criteria and alternatives

3.2 Sustainability Performance Assessment of Hydroelectric Power Plants with the T2NN-LOPCOW-CoCoSo Hybrid Method based on T2NNTWAM

The T2NN-LOPCOW-CoCoSo hybrid method based on T2NNTWAM is implemented in this part:

Step 1: Table 1 was employed to ascertain the levels of experience and expertise possessed by the experts, which were subsequently translated into T2NNs according to the details provided in the same table. The resulting T2NNs are given in Table A.1 (Appendix A).

Step 2: The aggregation of the experts' expertise and experience levels was conducted through the utilization of the T2NNTWAMA aggregation operator, as defined in Eq. (6). It was presumed that equal importance was attributed to the levels of expertise and experience exhibited by the experts. Thus, $\mathfrak{s}_1$ and $\mathfrak{s}_2$ were assumed to be 0.5, while the parameter is $\pi = 3.14$. The aggregation matrix is shown in Table A.2.

Step 3: The score $\mathcal{S}(\tilde{\text{Ⅎ}}_z)$ and accuracy $\mathcal{A}(\tilde{\text{Ⅎ}}_z)$ functions were computed. These values are presented in Table A.3.

Step 4: Utilizing Eq. (9), the corresponding vector ($\mathbb{E} = [\mathbb{E}_z]_{\sigma}$) for the experts was computed. This weighting is presented in Table 4.

Table 4
The weights of experts

Experts	$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_3$	$\mathcal{E}_4$	$\mathcal{E}_5$	$\mathcal{E}_6$	$\mathcal{E}_7$	$\mathcal{E}_8$	$\mathcal{E}_9$	$\mathcal{E}_{10}$
$\mathbb{E}_z$	0.0749	0.1093	0.0930	0.0930	0.1160	0.1253	0.0834	0.1027	0.1093	0.0930

Step 5: Using Table 2, initial decision matrices $\left(\tilde{\mathcal{F}}^{(z)} = \left[\tilde{\mathcal{F}}^{(z)}{}_{x\mathscr{y}^{(ql)}}\right]_{mxn^{(ql)}}\right)$ were constructed as LVs (Table A.4) and T2NNs (Table A.5).

Step 6: Using the T2NNTWAM shown in Eq. (6), the aggregated matrix $\left(\tilde{\mathcal{G}} = \left[\tilde{\mathcal{G}}_{x\mathscr{y}^{(ql)}}\right]_{mxn^{(ql)}}\right)$ was computed (Table A.6).

Step 7: The crisp matrix $\left(\mathcal{S}(\tilde{\mathcal{G}}) = \left[\mathcal{S}(\tilde{\mathcal{G}})_{x\mathscr{y}^{(ql)}}\right]_{mxn^{(ql)}}\right)$ was computed (Table A.7).

Step 8: The overall matrix $\left(\mathcal{H} = \left[\mathcal{H}_{x\mathscr{y}}\right]_{mxn}\right)$ was constructed (Table A.8).

Step 9: Using Eq. (10), Eq. (11), and Eq. (12), the normalized decision matrix $\left(\mathcal{H}^{(N)} = \left[\mathcal{H}^{(N)}{}_{x\mathscr{y}}\right]_{mxn}\right)$ was computed ($\varrho = 0.5$) (Table A.9).

Step 10: Using Eq. (13), Eq. (14), and Eq. (15), respectively, the root means square value matrix $\left(\mathcal{R} = \left[\mathcal{R}_{\mathscr{y}}\right]_n\right)$, the standard deviation matrix $\left(\mathcal{O} = \left[\mathcal{O}_{\mathscr{y}}\right]_n\right)$, and the contribution percentage of the criteria matrix $\left(\mathcal{U} = \left[\mathcal{U}_{\mathscr{y}}\right]_n\right)$ were computed (Table A.10).

Step 11: Using Eq. (16), the weighting matrix $\left(w = \left[w_{\mathscr{y}}\right]_n\right)$ of the criteria was computed. It is shown in Table 5.

Table 5
The weights of the criteria

Criteria	$\mathcal{C}_{1(qn)}$	$\mathcal{C}_{2(qn)}$	$\mathcal{C}_{3(qn)}$	$\mathcal{C}_{4(ql)}$	$\mathcal{C}_{5(ql)}$	$\mathcal{C}_{6(ql)}$	$\mathcal{C}_{7(ql)}$	$\mathcal{C}_{8(ql)}$	$\mathcal{C}_{9(ql)}$	$\mathcal{C}_{10(ql)}$	$\mathcal{C}_{11(ql)}$	$\mathcal{C}_{12(ql)}$
$w_{\mathscr{y}}$	0.06809	0.07583	0.07929	0.08457	0.09284	0.09383	0.09727	0.09774	0.07702	0.08063	0.07702	0.07586

Upon examining the criterion weights, it was observed that they range from 0.06809 to 0.09774, indicating a relatively close distribution. Since the approach is based on subjective expert evaluations, and the assessment focuses on HPPs within the same region with capacities ranging from 10 MW to

50 MW, it is expected that expert judgments would be similar. Therefore, although the criteria are appropriately defined, the closeness of the weights reflects the reliance on expert opinions.***Step 12:*** By performing Eq. (17), the normalized decision matrix $\left(\mathcal{K} = [\mathcal{K}_{xy}]_{mxn}\right)$ was calculated for alternative ranking (Table A.11).

Step 13: By performing Eq. (18), the cumulative value of the weighted comparability sequence matrix $(\mathbb{O} = [\mathbb{O}_x]_m)$ was computed (Table A.12).

Step 14: By performing Eq. (19), the combined influence of the power weighting in the comparability sequence matrix $(\mathbb{P} = [\mathbb{P}_x]_m)$ was computed (Table A.12).

Step 15: By using Eq. (20), Eq. (21), and Eq. (22), the relative importance of the alternatives through the utilization of scoring strategies for each alternative was computed $(\psi = 0.5)$ (Table A.13).

Step 16: By employed Eq. (23), the overall score value matrix $(\mathbb{R} = [\mathbb{R}_x]_m)$ was computed and presented in Table 6.

Table 6
The ranking of HPPs for the T2NNTWAM aggregation operator

	$\mathcal{H}_1$	$\mathcal{H}_2$	$\mathcal{H}_3$	$\mathcal{H}_4$	$\mathcal{H}_5$	$\mathcal{H}_6$	$\mathcal{H}_7$	$\mathcal{H}_8$	$\mathcal{H}_9$	$\mathcal{H}_{10}$	$\mathcal{H}_{11}$	$\mathcal{H}_{12}$	$\mathcal{H}_{13}$	$\mathcal{H}_{14}$	$\mathcal{H}_{15}$	$\mathcal{H}_{16}$	$\mathcal{H}_{17}$
$\mathbb{R}_x$	2.133	1.975	1.952	1.949	1.983	1.858	1.863	2.050	1.600	1.806	1.598	1.887	1.783	1.703	1.596	1.293	1.415
Rank	1	4	5	6	3	9	8	2	13	10	14	7	11	12	15	17	16

3.3 Sustainability Performance Assessment of Hydroelectric Power Plants with the T2NN-LOPCOW-CoCoSo Hybrid Method based on T2NNTWGM

Upon the application of the T2NNTWGM aggregation operator, identical procedural steps to those involving the T2NNTWAM aggregation operator were pursued. Nevertheless, during both Step 2 and Step 6, the T2NNYWGM aggregation operator is employed. The outcomes of this application are displayed in Table 7, showcasing alternative rankings.

Table 7
The ranking of HPPs for the T2NNTWGM aggregation operator

	$\mathcal{H}_1$	$\mathcal{H}_2$	$\mathcal{H}_3$	$\mathcal{H}_4$	$\mathcal{H}_5$	$\mathcal{H}_6$	$\mathcal{H}_7$	$\mathcal{H}_8$	$\mathcal{H}_9$	$\mathcal{H}_{10}$	$\mathcal{H}_{11}$	$\mathcal{H}_{12}$	$\mathcal{H}_{13}$	$\mathcal{H}_{14}$	$\mathcal{H}_{15}$	$\mathcal{H}_{16}$	$\mathcal{H}_{17}$
$\mathbb{R}_x$	1.975	1.830	1.874	1.864	1.866	1.773	1.771	1.847	1.510	1.763	1.622	1.366	1.699	1.719	1.692	1.275	1.506
Rank	1	6	2	4	3	7	8	5	14	9	13	16	11	10	12	17	15

4. Results and Implications

This study aims to evaluate the sustainability performance of HPPs through a novel approach, emphasizing a comprehensive assessment of their sustainability. For this purpose, the T2NN-LOPCOW-CoCoSo hybrid model is proposed. This model adopts a performance assessment approach based on both quantitative and qualitative parameters in place of the traditional approach. The neutrosophic sets [32], LOPCOW method [33], and CoCoSo method [34] used in this model, as in many other studies, have contributed to achieving successful results in this research as well. In this hybrid model, qualitative criteria are assessed for each HPP based on expert opinions, while quantitative criteria are determined based on the capacity and operational capacities of the HPPs. The proposed model is developed using T2NNs because it is better suited for the uncertainty and complexity of data obtained from expert opinions [35,36]. Hence, the problem is analyzed using neutrosophic logic.

The analysis of the case study results, which assessed the sustainability performance of HPPs in Artvin province, reveals the following ranking of criteria by importance: "adverse impact on local community health and safety" $\left(\mathcal{C}_{8(ql)}\right)$ > "adverse impact on the use of water resources" $\left(\mathcal{C}_{7(ql)}\right)$ >

"disaster risk" $\left(\mathcal{C}_{6(ql)}\right)$ > "adverse impact on cultural and historical heritage" $\left(\mathcal{C}_{5(ql)}\right)$ > "adverse impact on fish populations and behaviors" $\left(\mathcal{C}_{4(ql)}\right)$ > "contribution to the local economy" $\left(\mathcal{C}_{10(ql)}\right)$ > "the average annual production" $\left(\mathcal{C}_{3(qn)}\right)$ > "local community involvement" $\left(\mathcal{C}_{9(ql)}\right)$ > "contribution to transportation infrastructure" $\left(\mathcal{C}_{11(ql)}\right)$ > "waste management" $\left(\mathcal{C}_{12(ql)}\right)$ > "the annual production capacity" $\left(\mathcal{C}_{2(qn)}\right)$ > "installed capacity" $\left(\mathcal{C}_{1(qn)}\right)$. According to these results, the most important criterion group for evaluating the sustainability performance of HPPs is seen to be cost-oriented qualitative criteria. Within this group, the most critical criterion is "adverse impact on local community health and safety". In this context, the performance differences among HPPs can be largely explained by their scores on the health & safety criterion, which is considered the most critical factor in evaluating sustainability. Top-performing plants, such as Erenler and Çakırlar HPPs, achieve higher overall rankings primarily due to superior performance in this key criterion, highlighting the significant impact of human health and safety considerations within the sustainability framework.

Following this group, the importance ranking of criteria includes qualitative criteria with a benefit-oriented focus. However, there is one exception: "the average annual production" criterion. This criterion follows "contribution to the local economy" but ranks higher than other benefit-oriented qualitative criteria. Thus, it can be explained that, in the evaluation of HPPs' sustainability performance, the "the average annual production" criterion is more critical than some qualitative criteria. The other quantitative criteria are the least important, ranking lower in importance than all the qualitative criteria. At this point, it is evident that the criteria integrated into the decision model from social and environmental perspectives based on technical criteria are more critical in determining the sustainability performance of HPPs.

When examining the sustainability performance rankings of HPPs, using the T2NNTWAM aggregation operator in the T2NN-LOPCOW-CoCoSo hybrid model, the ranking is as follows: "Erenler HPP" $(\mathcal{H}_1)$ > "Çakırlar HPP" $(\mathcal{H}_8)$ > "Erenköy HPP" $(\mathcal{H}_5)$ > "Arpa HPP" $(\mathcal{H}_2)$ > "Artvin Kalecik HPP" $(\mathcal{H}_3)$ > "Papart HPP" $(\mathcal{H}_4)$ > "Şavşat HPP" $(\mathcal{H}_{12})$ > "Diyoban HPP" $(\mathcal{H}_7)$ > "Murgul HPP" $(\mathcal{H}_6)$ > "Orta HPP" $(\mathcal{H}_{10})$ > "Taşköprü HPP" $(\mathcal{H}_{13})$ > "Aralık HPP" $(\mathcal{H}_{14})$ > "Cala HPP" $(\mathcal{H}_9)$ > "Erik HPP" $(\mathcal{H}_{11})$ > "Cansu HPP" $(\mathcal{H}_{15})$ > "Kavak HPP" $(\mathcal{H}_{17})$ > "İskale HPP" $(\mathcal{H}_{16})$. According to these results, among HPPs with a capacity ranging from 10 MW to 50 MW in Artvin province, the top three HPPs with the highest sustainability performance are Erenler HPP, Çakırlar HPP, and Erenköy HPP, respectively.

When using the T2NNTWGM aggregation operator in the T2NN-LOPCOW-CoCoSo hybrid model, the ranking is as follows: $\mathcal{H}_1>\mathcal{H}_3>\mathcal{H}_5>\mathcal{H}_4>\mathcal{H}_8>\mathcal{H}_2>\mathcal{H}_6>\mathcal{H}_7>\mathcal{H}_{10}>\mathcal{H}_{14}>\mathcal{H}_{13}>\mathcal{H}_{15}>\mathcal{H}_{11}>\mathcal{H}_9>\mathcal{H}_{17}>\mathcal{H}_{12}>\mathcal{H}_{16}$. According to these results, among HPPs with capacities ranging from 10 MW to 50 MW in Artvin province, the top three HPPs with the highest sustainability performance are Erenler HPP, Artvin Kalecik HPP, and Erenköy HPP, respectively. Their superior performance is largely attributed to higher scores on the “Health & Safety” and other key sustainability criteria, which highlights the influence of these factors on overall rankings.

Comparing the model application results based on the T2NNTWAM and T2NNTWGM aggregation operators, in both methods, Erenler HPP is identified as the HPP with the highest sustainability performance. Thus, Erenler HPP is the best HPP in Artvin province in terms of sustainability performance.

4.1 Sensitivity Analysis

In this research, sensitivity analysis scenarios (SAS) were conducted for proposed T2NN-LOPCOW-CoCoSo model for determining the sustainable performance of HPPs, as well as the T2NNTWAM and T2NNTWGM aggregation operators developed for the model. In this section, two sensitivity analysis scenarios were created, each with a specific focus.

The first scenario investigates the impact of differences in the importance levels of two approaches used in the trigonometric operation-based-T2NN-LOPCOW method during the calculation of the normalized decision matrix (i.e., the max-min and sum normalization processes) on the overall results of the model (Table 8). The aim is to determine how variations in the effectiveness of these two approaches affect the ranking of HPPs.

The second scenario explores the differences in the importance levels of the relative importance levels of the alternatives in the trigonometric operation-based-T2NN-CoCoSo method. It examines the effect of changes in the contribution levels of arithmetic mean and sum in determining the final sustainability performance of HPPs.

For the first approach, SAS-1 was created, and ϱ values between 0 and 1 were assigned to determine the performance of HPPs. For the second approach, SAS-2 was created, and ψ values between 0 and 1 were assigned to determine the sustainability performance of HPPs. As a result, the differences in performance rankings obtained from the model application using both scenarios were identified. For SAS-1, the sustainability performance rankings of HPPs are shown in Table A.14.

For SAS-2, the sustainability performance rankings of HPPs for both T2NNTWAM and T2NNTWGM are shown in Table A.15. When examining the results of SAS-1 and SAS-2 scenarios, no significant changes are observed in the overall score values and performance rankings. In all scenarios, Erenler HPP was identified as the HPP with the highest sustainability performance. Thus, the research demonstrates that the proposed model provides robust results.

4.2 Research Implications

This research was conducted to develop a model for the determination of the sustainable performance of HPPs using the MCDM approach, in conjunction with T2NNs and novel aggregation operators (i.e., T2NNTWAM and T2NNTWGM). The findings and methodologies of this study could contribute to the creation of significant applications and decision-support tools in the sustainable energy sector.

Research implications and potential outcomes are:

i. *Decision support tools in the energy sector* – The model developed in this study can serve as a foundational tool for evaluating the sustainability performance of HPPs in the energy industry. This can aid energy companies and decision-makers in managing their resources more effectively.
ii. *Innovative aggregation operators* – The newly devised aggregation operators (T2NNTWAM and T2NNTWGM) may inspire other researchers for application in MCDM problems. These operators offer the ability to work with different types of neutrosophic data and conflicting criteria.
iii. *Contributions to sustainability studies* – The research contributes to the evaluation of HPPs from a sustainability perspective. This may facilitate the development of better policies and practices for sustainable energy production.
iv. *Application of neutrosophic sets and MCDM* – This study demonstrates the application of neutrosophic sets and MCDM techniques in the energy sector. This could inspire on how these techniques can be applied to other energy projects.
v. *Regional energy policies* – The study presents an exemplary case study of HPPs in the Artvin province of Türkiye. This can contribute to the formulation and implementation of local or regional energy policies.

4.3 Managerial Implications

This research contributes novel aggregation operators and by presenting a model for the assessment of HPP performance from a sustainability perspective. In light of these findings, several managerial implications emerge:

i. *Performance assessment tool* – The developed model enables managers and decision-makers to systematically evaluate the sustainability and operational performance of HPPs. For example, Erenler HPP's top ranking demonstrates how the model can identify high-performing plants and highlight areas for improvement in others, such as Çakırlar HPP.
ii. *Resource optimization* – By pinpointing strengths and weaknesses in specific criteria, such as "Health & Safety," managers can allocate resources more effectively. In Artvin, this could involve prioritizing maintenance or safety measures in lower-performing plants to enhance overall efficiency.
iii. *Investment decisions* – The model provides a structured approach to assess potential new HPP projects. Authorities can evaluate expected sustainability outcomes, ensuring investments target projects with higher environmental and social performance.
iv. *Policy development* – Policymakers can use the insights to design region-specific regulations or incentives. For instance, supporting HPPs that score well in environmental and social criteria could guide sustainable regional energy planning in Artvin.
v. *Risk mitigation* – Sensitivity analyses highlight potential variations in plant performance. Managers can proactively plan for uncertainties, such as fluctuations in output or environmental impacts, to ensure a stable energy supply.
vi. *Benchmarking and best practices* – Operators can compare their plants against top performers like Erenler HPP to identify gaps and implement strategies that align with or surpass industry standards. This promotes continuous improvement and sustainable practices.

This research offers practical tools and insights that can benefit energy sector managers, policymakers, and industry professionals. By employing the model and aggregation operators, stakeholders can make more informed decisions, optimize resource allocation, and contribute to the sustainable operation of HPPs.

5. Conclusions

This study present a comprehensive model for assessing the sustainable performance of HPPs by integrating a MCDM framework with T2NNs. A key contribution of the research is the introduction of two novel aggregation operators, T2NNTWAM and T2NNTWGM, based on trigonometric t-norm and t-conorm operations. These operators enhance the model's ability to handle uncertainty and complexity, which are inherent in sustainability evaluations and operational performance assessments.

The model was applied to a case study of HPPs in the Artvin province of Türkiye, with installed capacities ranging from 10 MW to 50 MW. The results demonstrated the model's effectiveness in providing a structured evaluation of sustainable performance, while sensitivity analyses confirmed its robustness under varying conditions. This illustrates the model's capacity to support reliable decision-making even when confronted with uncertain or incomplete information.

From a theoretical perspective, this paper presents a structured and data-driven framework that integrates both operational and sustainability dimensions of HPPs to the literature. The model also functions as a benchmarking tool, allowing operators to identify strengths and weaknesses in their current practices and align them with established industry standards and sustainability objectives.

This dual focus on evaluation and benchmarking provides a meaningful contribution to research on energy sustainability and performance management.

Practically, the model supports decision-makers in key areas such as resource allocation, investment planning, and policy development. By providing quantitative and systematic insights into HPP performance, it facilitates more informed, transparent, and resilient management strategies. The incorporation of sensitivity analyses further strengthens the model's applicability by highlighting potential risks and uncertainties, thereby promoting adaptive and forward-looking operational planning.

Several avenues for future research emerge from this study. The model and aggregation operators can be extended to other renewable energy sources, such as wind, solar, or nuclear power, enabling comparative analyses and cross-sector insights. Long-term monitoring of HPP performance, cross-national studies, and evaluations of policy or regulatory impacts could further enhance understanding of sustainability dynamics. Additionally, assessing the resilience of HPPs to climate-related risks and extreme events will be increasingly important for ensuring sustainable and reliable energy supply. Such research directions will support the continuous advancement of sustainable energy solutions and the integration of HPPs within a resilient, diversified, and globally sustainable energy portfolio.

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Conflicts of Interest
The authors declare no conflicts of interest.

Declaration on the Use of Artificial Intelligence Tools
During the preparation of this work the authors used Grammarly and DeepL solely for language editing and proofreading. After using this tool, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

Appendix A: Supplementary Tables

Table A.1
The experience and expertise levels of 10 experts

Experts	Experience	Expertise	T2NNs	
			Experience (years)	Expertise
$\mathcal{E}_1$	11	MG	⟨(0.4,0.3,0.35),(0.5,0.45,0.5),(0.45,0.4,0.60)⟩	⟨(0.6,0.45,0.5),(0.2,0.15,0.25),(0.1,0.25,0.15)⟩
$\mathcal{E}_2$	14	MG	⟨(0.4,0.3,0.35),(0.5,0.45,0.5),(0.45,0.4,0.60)⟩	⟨(0.6,0.45,0.5),(0.2,0.15,0.25),(0.1,0.25,0.15)⟩
$\mathcal{E}_3$	18	G	⟨(0.5,0.45,0.5),(0.4,0.35,0.5),(0.35,0.3,0.45)⟩	⟨(0.7,0.75,0.8),(0.15,0.2,0.25),(0.1,0.15,0.2)⟩
$\mathcal{E}_4$	21	G	⟨(0.6,0.45,0.5),(0.2,0.15,0.25),(0.1,0.25,0.15)⟩	⟨(0.7,0.75,0.8),(0.15,0.2,0.25),(0.1,0.15,0.2)⟩
$\mathcal{E}_5$	25	VG	⟨(0.7,0.75,0.8),(0.15,0.2,0.25),(0.1,0.15,0.2)⟩	⟨(0.95,0.9,0.95),(0.1,0.1,0.05),(0.05,0.05,0.05)⟩
$\mathcal{E}_6$	24	VG	⟨(0.6,0.45,0.5),(0.2,0.15,0.25),(0.1,0.25,0.15)⟩	⟨(0.95,0.9,0.95),(0.1,0.1,0.05),(0.05,0.05,0.05)⟩
$\mathcal{E}_7$	20	G	⟨(0.6,0.45,0.5),(0.2,0.15,0.25),(0.1,0.25,0.15)⟩	⟨(0.7,0.75,0.8),(0.15,0.2,0.25),(0.1,0.15,0.2)⟩
$\mathcal{E}_8$	17	MG	⟨(0.5,0.45,0.5),(0.4,0.35,0.5),(0.35,0.3,0.45)⟩	⟨(0.6,0.45,0.5),(0.2,0.15,0.25),(0.1,0.25,0.15)⟩
$\mathcal{E}_9$	22	G	⟨(0.6,0.45,0.5),(0.2,0.15,0.25),(0.1,0.25,0.15)⟩	⟨(0.7,0.75,0.8),(0.15,0.2,0.25),(0.1,0.15,0.2)⟩
$\mathcal{E}_{10}$	15	MG	⟨(0.5,0.45,0.5),(0.4,0.35,0.5),(0.35,0.3,0.45)⟩	⟨(0.6,0.45,0.5),(0.2,0.15,0.25),(0.1,0.25,0.15)⟩

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Table A.2
The experience and expertise levels of 10 experts

Experts	$\tilde{T}$			$\tilde{I}$			$\tilde{F}$		
	T	I	F	T	I	F	T	I	F
$\mathcal{E}_1$	0.516	0.381	0.432	0.290	0.228	0.361	0.166	0.309	0.247
$\mathcal{E}_2$	0.516	0.381	0.432	0.290	0.228	0.361	0.166	0.309	0.247
$\mathcal{E}_3$	0.622	0.650	0.710	0.220	0.256	0.337	0.157	0.201	0.280
$\mathcal{E}_4$	0.656	0.650	0.710	0.172	0.172	0.250	0.100	0.188	0.172
$\mathcal{E}_5$	0.913	0.856	0.920	0.120	0.134	0.084	0.067	0.075	0.080
$\mathcal{E}_6$	0.910	0.827	0.907	0.134	0.120	0.084	0.067	0.084	0.075
$\mathcal{E}_7$	0.656	0.650	0.710	0.172	0.172	0.250	0.100	0.188	0.172
$\mathcal{E}_8$	0.555	0.450	0.500	0.269	0.211	0.337	0.157	0.273	0.228
$\mathcal{E}_9$	0.656	0.650	0.710	0.172	0.172	0.250	0.100	0.188	0.172
$\mathcal{E}_{10}$	0.555	0.450	0.500	0.269	0.211	0.337	0.157	0.273	0.228

Table A.3
The functions values for expert weights

Experts	$\mathcal{S}(\tilde{Ⅎ}_z)$	$\mathcal{A}(\tilde{Ⅎ}_z)$
$\mathcal{E}_1$	0.6308	0.1696
$\mathcal{E}_2$	0.6308	0.1696
$\mathcal{E}_3$	0.7270	0.4484
$\mathcal{E}_4$	0.7712	0.5049
$\mathcal{E}_5$	0.8981	0.8122
$\mathcal{E}_6$	0.8919	0.7903
$\mathcal{E}_7$	0.7712	0.5049
$\mathcal{E}_8$	0.6662	0.2559
$\mathcal{E}_9$	0.7712	0.5049
$\mathcal{E}_{10}$	0.6662	0.2559

Table A.4
The matrices with LVs for qualitative criteria

Alter.	Expert	$\mathcal{C}_{4(ql)}$	$\mathcal{C}_{5(ql)}$	$\mathcal{C}_{6(ql)}$	$\mathcal{C}_{7(ql)}$	$\mathcal{C}_{8(ql)}$	$\mathcal{C}_{9(ql)}$	$\mathcal{C}_{10(ql)}$	$\mathcal{C}_{11(ql)}$	$\mathcal{C}_{12(ql)}$
$\mathcal{H}_1$	$\mathcal{E}_1$	VH	ML	VL	ML	L	ML	L	L	L
	$\mathcal{E}_2$	VH	M	VL	ML	L	MH	L	VL	L
	$\mathcal{E}_3$	H	M	VL	M	L	M	L	L	L
	$\mathcal{E}_4$	MH	ML	VL	ML	L	M	M	VL	VL
	$\mathcal{E}_5$	MH	M	L	L	VL	M	L	VL	M
	$\mathcal{E}_6$	H	M	VL	VL	L	ML	M	L	L
	$\mathcal{E}_7$	VH	ML	VL	L	ML	VL	VL	ML	ML
	$\mathcal{E}_8$	MH	ML	L	ML	ML	ML	L	M	L
	$\mathcal{E}_9$	VH	ML	L	L	ML	ML	L	VL	L
	$\mathcal{E}_{10}$	VH	ML	VL	ML	L	ML	L	L	VL
⋮										
	Expert	$\mathcal{C}_{4(ql)}$	$\mathcal{C}_{5(ql)}$	$\mathcal{C}_{6(ql)}$	$\mathcal{C}_{7(ql)}$	$\mathcal{C}_{8(ql)}$	$\mathcal{C}_{9(ql)}$	$\mathcal{C}_{10(ql)}$	$\mathcal{C}_{11(ql)}$	$\mathcal{C}_{12(ql)}$
$\mathcal{H}_{17}$	$\mathcal{E}_1$	H	M	ML	ML	L	VL	ML	ML	ML
	$\mathcal{E}_2$	H	ML	MH	M	VL	L	M	ML	ML
	$\mathcal{E}_3$	H	MH	MH	M	L	VL	M	M	M
	$\mathcal{E}_4$	H	ML	VL	MH	VL	VL	ML	M	ML
	$\mathcal{E}_5$	H	M	VL	ML	VL	L	ML	M	M
	$\mathcal{E}_6$	VH	M	VL	VL	M	VL	ML	L	M
	$\mathcal{E}_7$	MH	M	VL	VL	L	L	VL	L	ML
	$\mathcal{E}_8$	H	L	L	L	L	L	MH	L	ML
	$\mathcal{E}_9$	H	M	ML	ML	L	VL	VL	ML	L
	$\mathcal{E}_{10}$	MH	MH	ML	L	L	VL	M	ML	L

Table A.5
The matrices with LVs for qualitative criteria*

	Expert	$\mathcal{C}_{4(ql)}$								
		$T_{T_{\tilde{\mathfrak{J}}_g}}(\hbar)$	$T_{I_{\tilde{\mathfrak{J}}_g}}(\hbar)$	$T_{F_{\tilde{\mathfrak{J}}_g}}(\hbar)$	$I_{T_{\tilde{\mathfrak{J}}_g}}(\hbar)$	$I_{I_{\tilde{\mathfrak{J}}_g}}(\hbar)$	$I_{F_{\tilde{\mathfrak{J}}_g}}(\hbar)$	$F_{T_{\tilde{\mathfrak{J}}_g}}(\hbar)$	$F_{I_{\tilde{\mathfrak{J}}_g}}(\hbar)$	$F_{F_{\tilde{\mathfrak{J}}_g}}(\hbar)$
$\mathcal{H}_1$	$\mathcal{E}_1$	0.95	0.90	0.95	0.10	0.10	0.05	0.05	0.05	0.05
	$\mathcal{E}_2$	0.95	0.90	0.95	0.10	0.10	0.05	0.05	0.05	0.05
	$\mathcal{E}_3$	0.70	0.75	0.80	0.15	0.20	0.25	0.10	0.15	0.20
	$\mathcal{E}_4$	0.60	0.45	0.50	0.20	0.15	0.25	0.10	0.25	0.15
	$\mathcal{E}_5$	0.60	0.45	0.50	0.20	0.15	0.25	0.10	0.25	0.15
	$\mathcal{E}_6$	0.70	0.75	0.80	0.15	0.20	0.25	0.10	0.15	0.20
	$\mathcal{E}_7$	0.95	0.90	0.95	0.10	0.10	0.05	0.05	0.05	0.05
	$\mathcal{E}_8$	0.60	0.45	0.50	0.20	0.15	0.25	0.10	0.25	0.15
	$\mathcal{E}_9$	0.95	0.90	0.95	0.10	0.10	0.05	0.05	0.05	0.05
	$\mathcal{E}_{10}$	0.95	0.90	0.95	0.10	0.10	0.05	0.05	0.05	0.05
						$\vdots$				
	Expert	$\mathcal{C}_{12(ql)}$								
		$T_{T_{\tilde{\mathfrak{J}}_g}}(\hbar)$	$T_{I_{\tilde{\mathfrak{J}}_g}}(\hbar)$	$T_{F_{\tilde{\mathfrak{J}}_g}}(\hbar)$	$T_{T_{\tilde{\mathfrak{J}}_g}}(\hbar)$	$I_{I_{\tilde{\mathfrak{J}}_g}}(\hbar)$	$I_{F_{\tilde{\mathfrak{J}}_g}}(\hbar)$	$T_{T_{\tilde{\mathfrak{J}}_g}}(\hbar)$	$F_{I_{\tilde{\mathfrak{J}}_g}}(\hbar)$	$F_{F_{\tilde{\mathfrak{J}}_g}}(\hbar)$
$\mathcal{H}_1$	$\mathcal{E}_1$	0.35	$\mathcal{H}_1$	$\mathcal{E}_1$	0.35	$\mathcal{H}_1$	$\mathcal{E}_1$	0.35	$\mathcal{H}_1$	$\mathcal{E}_1$
	$\mathcal{E}_2$	0.35	0.45	$\mathcal{E}_2$	0.35	0.35	$\mathcal{E}_2$	0.35	0.30	$\mathcal{E}_2$
	$\mathcal{E}_3$	0.35	0.45	$\mathcal{E}_3$	0.35	0.35	$\mathcal{E}_3$	0.35	0.30	$\mathcal{E}_3$
	$\mathcal{E}_4$	0.20	0.30	$\mathcal{E}_4$	0.20	0.45	$\mathcal{E}_4$	0.20	0.40	$\mathcal{E}_4$
	$\mathcal{E}_5$	0.50	0.45	$\mathcal{E}_5$	0.50	0.35	$\mathcal{E}_5$	0.50	0.30	$\mathcal{E}_5$
	$\mathcal{E}_6$	0.35	0.45	$\mathcal{E}_6$	0.35	0.35	$\mathcal{E}_6$	0.35	0.30	$\mathcal{E}_6$
	$\mathcal{E}_7$	0.40	0.30	$\mathcal{E}_7$	0.40	0.45	$\mathcal{E}_7$	0.40	0.40	$\mathcal{E}_7$
	$\mathcal{E}_8$	0.35	0.30	$\mathcal{E}_8$	0.35	0.45	$\mathcal{E}_8$	0.35	0.40	$\mathcal{E}_8$
	$\mathcal{E}_9$	0.35	0.30	$\mathcal{E}_9$	0.35	0.45	$\mathcal{E}_9$	0.35	0.40	$\mathcal{E}_9$
	$\mathcal{E}_{10}$	0.20	0.30	$\mathcal{E}_{10}$	0.20	0.45	$\mathcal{E}_{10}$	0.20	0.40	$\mathcal{E}_{10}$

* This is shown for only the first alternative. The same procedure was applied to the other alternatives as well

Table A.6
The aggregated decision matrix

Alternatives	$\mathcal{C}_{4(ql)}$								
	$T_{T_{\tilde{\mathfrak{J}}}}(\hbar)$	$T_{I_{\tilde{\mathfrak{J}}}}(\hbar)$	$T_{F_{\tilde{\mathfrak{J}}}}(\hbar)$	$I_{T_{\tilde{\mathfrak{J}}}}(\hbar)$	$I_{I_{\tilde{\mathfrak{J}}}}(\hbar)$	$I_{F_{\tilde{\mathfrak{J}}}}(\hbar)$	$F_{T_{\tilde{\mathfrak{J}}}}(\hbar)$	$F_{I_{\tilde{\mathfrak{J}}}}(\hbar)$	$F_{F_{\tilde{\mathfrak{J}}}}(\hbar)$
$\mathcal{H}_1$	0.9126	0.8678	0.9208	0.0950	0.1035	0.0806	0.0537	0.0689	0.0719
$\mathcal{H}_2$	0.9209	0.8722	0.9263	0.0984	0.1052	0.0748	0.0550	0.0657	0.0691
$\mathcal{H}_3$	0.8954	0.8528	0.9060	0.0935	0.0977	0.0916	0.0532	0.0757	0.0764
$\mathcal{H}_4$	0.9520	0.9130	0.9533	0.0726	0.0733	0.0463	0.0382	0.0436	0.0436
$\mathcal{H}_5$	0.9447	0.9079	0.9474	0.0670	0.0679	0.0511	0.0361	0.0464	0.0455
$\mathcal{H}_6$	0.9511	0.9212	0.9552	0.0623	0.0694	0.0468	0.0348	0.0411	0.0438
$\mathcal{H}_7$	0.9194	0.8900	0.9312	0.0779	0.0917	0.0742	0.0460	0.0592	0.0664
$\mathcal{H}_8$	0.9031	0.8526	0.9099	0.0966	0.0980	0.0858	0.0533	0.0748	0.0726
$\mathcal{H}_9$	0.9349	0.8899	0.9374	0.0808	0.0805	0.0603	0.0434	0.0550	0.0538
$\vdots$									
$\mathcal{H}_{17}$	0.9681	0.9488	0.9706	0.0392	0.0427	0.0304	0.0219	0.0265	0.0279
Alternatives	$\mathcal{C}_{12(ql)}$								
	$T_{T_{\tilde{\mathfrak{J}}}}(\hbar)$	$T_{I_{\tilde{\mathfrak{J}}}}(\hbar)$	$T_{T_{\tilde{\mathfrak{J}}}}(\hbar)$	$I_{T_{\tilde{\mathfrak{J}}}}(\hbar)$	$T_{T_{\tilde{\mathfrak{J}}}}(\hbar)$	$I_{F_{\tilde{\mathfrak{J}}}}(\hbar)$	$T_{T_{\tilde{\mathfrak{J}}}}(\hbar)$	$F_{I_{\tilde{\mathfrak{J}}}}(\hbar)$	$T_{T_{\tilde{\mathfrak{J}}}}(\hbar)$
$\mathcal{H}_1$	0.4925	0.4119	0.3854	0.3924	0.4031	0.5435	0.3497	0.3607	0.5024
$\mathcal{H}_2$	0.4958	0.4228	0.3805	0.3639	0.3719	0.5144	0.2818	0.3683	0.4208
$\mathcal{H}_3$	0.5589	0.4927	0.4834	0.3326	0.3318	0.4687	0.2854	0.2911	0.4156
$\mathcal{H}_4$	0.5637	0.5011	0.4392	0.3031	0.3201	0.4542	0.2330	0.3179	0.3545
$\mathcal{H}_5$	0.6333	0.5764	0.5331	0.2517	0.2580	0.3802	0.1957	0.2510	0.2946
$\mathcal{H}_6$	0.5977	0.5267	0.4905	0.2958	0.3112	0.4404	0.2547	0.2753	0.3883
$\mathcal{H}_7$	0.6049	0.5248	0.5421	0.2832	0.2586	0.3868	0.2214	0.2446	0.3260
$\mathcal{H}_8$	0.5509	0.4837	0.4514	0.3044	0.2965	0.4334	0.2174	0.3129	0.3250
$\mathcal{H}_9$	0.5679	0.5155	0.4365	0.2979	0.3247	0.4578	0.2306	0.3237	0.3485
$\vdots$									
$\mathcal{H}_{17}$	0.7620	0.7090	0.6922	0.1585	0.1551	0.2406	0.1199	0.1501	0.1868

Table A.7
The crisp initial decision matrix for qualitative criteria

Alternatives	$\mathcal{C}_{4(ql)}$	$\mathcal{C}_{5(ql)}$	$\mathcal{C}_{6(ql)}$	$\mathcal{C}_{7(ql)}$	$\mathcal{C}_{8(ql)}$	$\mathcal{C}_{9(ql)}$	$\mathcal{C}_{10(ql)}$	$\mathcal{C}_{11(ql)}$	$\mathcal{C}_{12(ql)}$
$\mathcal{H}_1$	0.9102	0.4724	0.4174	0.7166	0.4833	0.5312	0.4676	0.4746	0.5322
$\mathcal{H}_1$	0.9127	0.5068	0.5307	0.7665	0.5415	0.6116	0.5601	0.5443	0.5550
$\mathcal{H}_2$	0.9038	0.5836	0.5872	0.7766	0.6238	0.5955	0.5794	0.5655	0.6067
$\mathcal{H}_3$	0.9414	0.5777	0.5440	0.7935	0.5916	0.6407	0.5551	0.5709	0.6153
$\mathcal{H}_4$	0.9400	0.6261	0.5934	0.7787	0.5790	0.6775	0.6404	0.6308	0.6816
$\mathcal{H}_5$	0.9450	0.6398	0.5439	0.7124	0.6753	0.6938	0.5522	0.5914	0.6324
$\mathcal{H}_6$	0.9220	0.6051	0.4719	0.7389	0.5703	0.6146	0.5353	0.5170	0.6644
$\mathcal{H}_7$	0.9054	0.4967	0.4602	0.6495	0.5406	0.5469	0.5647	0.5778	0.6226
$\mathcal{H}_8$	0.9286	0.5540	0.5162	0.7676	0.5752	0.6446	0.5405	0.5440	0.6170
$\mathcal{H}_9$	0.9054	0.5190	0.4639	0.7191	0.5094	0.5780	0.4960	0.5251	0.5160
$\mathcal{H}_{10}$	0.8991	0.4455	0.4051	0.7078	0.4689	0.5101	0.4525	0.4597	0.5191
$\mathcal{H}_{11}$	0.9024	0.4721	0.5219	0.7605	0.5294	0.5679	0.5483	0.5355	0.5432
$\mathcal{H}_{12}$	0.8991	0.5531	0.5791	0.7670	0.6135	0.5669	0.5669	0.5525	0.5952
$\mathcal{H}_{13}$	0.9640	0.7130	0.6817	0.8736	0.7264	0.7573	0.6939	0.7158	0.7542
$\mathcal{H}_{14}$	0.9638	0.7410	0.7421	0.8805	0.7416	0.7973	0.7723	0.7646	0.7818
$\mathcal{H}_{15}$	0.9643	0.7711	0.7460	0.8679	0.8055	0.8028	0.7358	0.7542	0.7833
$\mathcal{H}_{16}$	0.9649	0.7613	0.7013	0.8791	0.7527	0.7891	0.7304	0.7248	0.7963
$\mathcal{H}_{17}$	0.9127	0.5068	0.5307	0.7665	0.5415	0.6116	0.5601	0.5443	0.5550

Table A.8
The initial decision matrix for total criteria

Alternatives	$\mathcal{C}_{1(qn)}$	$\mathcal{C}_{2(qn)}$	$\mathcal{C}_{3(qn)}$	$\mathcal{C}_{4(ql)}$	$\mathcal{C}_{5(ql)}$	$\mathcal{C}_{6(ql)}$	$\mathcal{C}_{7(ql)}$	$\mathcal{C}_{8(ql)}$	$\mathcal{C}_{9(ql)}$	$\mathcal{C}_{10(ql)}$	$\mathcal{C}_{11(ql)}$	$\mathcal{C}_{12(ql)}$
$\mathcal{H}_1$	45.00	125.50	96.00	0.9102	0.4724	0.4174	0.7166	0.4833	0.5312	0.4676	0.4746	0.5322
$\mathcal{H}_1$	32.41	77.66	43.00	0.9127	0.5068	0.5307	0.7665	0.5415	0.6116	0.5601	0.5443	0.5550
$\mathcal{H}_2$	27.50	71.15	71.00	0.9038	0.5836	0.5872	0.7766	0.6238	0.5955	0.5794	0.5655	0.6067
$\mathcal{H}_3$	26.60	106.00	68.00	0.9414	0.5777	0.5440	0.7935	0.5916	0.6407	0.5551	0.5709	0.6153
$\mathcal{H}_4$	21.46	86.97	64.00	0.9400	0.6261	0.5934	0.7787	0.5790	0.6775	0.6404	0.6308	0.6816
$\mathcal{H}_5$	19.60	50.00	55.00	0.9450	0.6398	0.5439	0.7124	0.6753	0.6938	0.5522	0.5914	0.6324
$\mathcal{H}_6$	19.04	35.79	34.00	0.9220	0.6051	0.4719	0.7389	0.5703	0.6146	0.5353	0.5170	0.6644
$\mathcal{H}_7$	16.21	60.00	48.00	0.9054	0.4967	0.4602	0.6495	0.5406	0.5469	0.5647	0.5778	0.6226
$\mathcal{H}_8$	15.61	22.00	21.00	0.9286	0.5540	0.5162	0.7676	0.5752	0.6446	0.5405	0.5440	0.6170
$\mathcal{H}_9$	15.36	55.24	41.00	0.9054	0.5190	0.4639	0.7191	0.5094	0.5780	0.4960	0.5251	0.5160
$\vdots$												
$\mathcal{H}_{17}$	45.00	125.50	96.00	0.9127	0.5068	0.5307	0.7665	0.5415	0.6116	0.5601	0.5443	0.5550

Table A.9
The normalized decision matrix

Alternatives	$\mathcal{C}_{1(qn)}$	$\mathcal{C}_{2(qn)}$	$\mathcal{C}_{3(qn)}$	$\mathcal{C}_{4(ql)}$	$\mathcal{C}_{5(ql)}$	$\mathcal{C}_{6(ql)}$	$\mathcal{C}_{7(ql)}$	$\mathcal{C}_{8(ql)}$	$\mathcal{C}_{9(ql)}$	$\mathcal{C}_{10(ql)}$	$\mathcal{C}_{11(ql)}$	$\mathcal{C}_{12(ql)}$
$\mathcal{H}_1$	0.5693	0.5633	0.5611	0.4451	0.4945	0.5201	0.3863	0.5147	0.0604	0.0470	0.0481	0.0534
$\mathcal{H}_1$	0.3691	0.3081	0.1740	0.4263	0.4392	0.3459	0.2763	0.4243	0.2014	0.1962	0.1658	0.0952
$\mathcal{H}_2$	0.2910	0.2733	0.3785	0.4944	0.3168	0.2600	0.2539	0.2978	0.1733	0.2274	0.2017	0.1897
$\mathcal{H}_3$	0.2767	0.4593	0.3566	0.2074	0.3263	0.3256	0.2166	0.3472	0.2525	0.1882	0.2108	0.2056
$\mathcal{H}_4$	0.1950	0.3578	0.3274	0.2183	0.2497	0.2506	0.2494	0.3666	0.3170	0.3258	0.3119	0.3269
$\mathcal{H}_5$	0.1654	0.1605	0.2617	0.1798	0.2280	0.3258	0.3956	0.2192	0.3456	0.1835	0.2454	0.2369
$\mathcal{H}_6$	0.1565	0.0847	0.1083	0.3553	0.2828	0.4358	0.3370	0.3800	0.2067	0.1562	0.1197	0.2954
$\mathcal{H}_7$	0.1115	0.2139	0.2105	0.4824	0.4553	0.4538	0.5348	0.4258	0.0879	0.2036	0.2225	0.2188
$\mathcal{H}_8$	0.1020	0.0111	0.0134	0.3053	0.3639	0.3679	0.2738	0.3725	0.2594	0.1647	0.1653	0.2086
$\mathcal{H}_9$	0.0980	0.1885	0.1594	0.4824	0.4197	0.4481	0.3808	0.4740	0.1424	0.0927	0.1334	0.0239
$\vdots$												
$\mathcal{H}_{17}$	0.5693	0.5633	0.5611	0.4451	0.4945	0.5201	0.3863	0.5147	0.0604	0.0470	0.0481	0.0534

Table A.10
The root means square value, the standard deviation, and the contribution percentage of criteria matrices

Alternatives	$\mathcal{C}_{1(qn)}$	$\mathcal{C}_{2(qn)}$	$\mathcal{C}_{3(qn)}$	$\mathcal{C}_{4(ql)}$	$\mathcal{C}_{5(ql)}$	$\mathcal{C}_{6(ql)}$	$\mathcal{C}_{7(ql)}$	$\mathcal{C}_{8(ql)}$	$\mathcal{C}_{9(ql)}$	$\mathcal{C}_{10(ql)}$	$\mathcal{C}_{11(ql)}$	$\mathcal{C}_{12(ql)}$
$\mathcal{R}_{y}$	0.212329	0.249672	0.243108	0.364139	0.346545	0.341406	0.297564	0.355937	0.303402	0.282456	0.289657	0.296078
$\mathcal{O}_{y}$	0.081145	0.085536	0.07932	0.110262	0.093374	0.090703	0.075308	0.089482	0.102211	0.090433	0.097583	0.1014
$\mathcal{U}_{y}$	0.961898	1.071213	1.120016	1.194679	1.311399	1.325486	1.374045	1.380717	1.088024	1.138919	1.087993	1.071551

Table A.11
The normalized decision matrix for alternative ranking

Alternatives	$\mathcal{C}_{1(qn)}$	$\mathcal{C}_{2(qn)}$	$\mathcal{C}_{3(qn)}$	$\mathcal{C}_{4(ql)}$	$\mathcal{C}_{5(ql)}$	$\mathcal{C}_{6(ql)}$	$\mathcal{C}_{7(ql)}$	$\mathcal{C}_{8(ql)}$	$\mathcal{C}_{9(ql)}$	$\mathcal{C}_{10(ql)}$	$\mathcal{C}_{11(ql)}$	$\mathcal{C}_{12(ql)}$
$\mathcal{H}_{1}$	1.0000	1.0000	1.0000	0.8303	0.9175	0.9640	0.7096	0.9572	0.0723	0.0472	0.0490	0.0576
$\mathcal{H}_{1}$	0.6384	0.5378	0.2933	0.7929	0.8119	0.6317	0.4936	0.7842	0.3468	0.3363	0.2774	0.1392
$\mathcal{H}_{2}$	0.4974	0.4749	0.6667	0.9285	0.5758	0.4658	0.4495	0.5397	0.2921	0.3969	0.3471	0.3233
$\mathcal{H}_{3}$	0.4716	0.8116	0.6267	0.3568	0.5941	0.5926	0.3763	0.6354	0.4463	0.3209	0.3648	0.3543
$\mathcal{H}_{4}$	0.3240	0.6277	0.5733	0.3785	0.4454	0.4476	0.4408	0.6729	0.5720	0.5876	0.5611	0.5907
$\mathcal{H}_{5}$	0.2705	0.2705	0.4533	0.3019	0.4032	0.5931	0.7277	0.3868	0.6277	0.3116	0.4319	0.4154
$\mathcal{H}_{6}$	0.2545	0.1332	0.1733	0.6514	0.5098	0.8041	0.6129	0.6988	0.3572	0.2588	0.1879	0.5294
$\mathcal{H}_{7}$	0.1732	0.3671	0.3600	0.9045	0.8427	0.8384	1.0000	0.7870	0.1257	0.3507	0.3874	0.3801
$\mathcal{H}_{8}$	0.1559	0.0000	0.0000	0.5518	0.6669	0.6742	0.4887	0.6843	0.4597	0.2752	0.2766	0.3602
$\mathcal{H}_{9}$	0.1488	0.3212	0.2667	0.9045	0.7744	0.8275	0.6988	0.8796	0.2320	0.1358	0.2146	0.0000
$\mathcal{H}_{10}$	0.1390	0.1862	0.2667	0.9998	1.0000	1.0000	0.7476	1.0000	0.0000	0.0000	0.0000	0.0109
$\mathcal{H}_{11}$	0.1246	0.1836	0.3200	0.9501	0.9185	0.6575	0.5194	0.8202	0.1975	0.2996	0.2486	0.0970
$\mathcal{H}_{12}$	0.0643	0.2025	0.0400	1.0000	0.6695	0.4895	0.4912	0.5704	0.1943	0.3578	0.3044	0.2826
$\mathcal{H}_{13}$	0.0640	0.3285	0.3600	0.0131	0.1785	0.1888	0.0296	0.2351	0.8447	0.7548	0.8399	0.8498
$\mathcal{H}_{14}$	0.0235	0.2857	0.2533	0.0169	0.0923	0.0114	0.0000	0.1897	0.9812	1.0000	1.0000	0.9483
$\mathcal{H}_{15}$	0.0092	0.0556	0.0533	0.0081	0.0000	0.0000	0.0545	0.0000	1.0000	0.8860	0.9656	0.9535
$\mathcal{H}_{16}$	0.0000	0.1736	0.0133	0.0000	0.0302	0.1313	0.0059	0.1567	0.9533	0.8690	0.8692	1.0000
$\mathcal{H}_{17}$	1.0000	1.0000	1.0000	0.8303	0.9175	0.9640	0.7096	0.9572	0.0723	0.0472	0.0490	0.0576

Table A.12
The cumulative value and the combined influence matrices

Alternatives	$\mathbb{O}_{x}$	$\mathbb{P}_{x}$
$\mathcal{H}_{1}$	0.6492	11.1327
$\mathcal{H}_{1}$	0.5196	11.2652
$\mathcal{H}_{2}$	0.5007	11.2857
$\mathcal{H}_{3}$	0.4982	11.2872
$\mathcal{H}_{4}$	0.5186	11.3431
$\mathcal{H}_{5}$	0.4403	11.1635
$\mathcal{H}_{6}$	0.4500	11.0947
$\mathcal{H}_{7}$	0.5724	11.3029
$\mathcal{H}_{8}$	0.4031	9.3295
$\mathcal{H}_{9}$	0.4810	10.2332
$\mathcal{H}_{10}$	0.4872	8.3371
$\mathcal{H}_{11}$	0.4717	11.0552
$\mathcal{H}_{12}$	0.4078	10.8917
$\mathcal{H}_{13}$	0.3741	10.5883
$\mathcal{H}_{14}$	0.3781	9.5925
$\mathcal{H}_{15}$	0.3102	7.7252
$\mathcal{H}_{16}$	0.3315	8.5502
$\mathcal{H}_{17}$	0.6492	11.1327

Table A.13
The relative importance of alternatives through the utilization of scoring strategies for each alternative

Alternatives	$\mathbb{R}_{xa}$	$\mathbb{R}_{xb}$	$\mathbb{R}_{xc}$
$\mathcal{H}_1$	0.064042	3.533776	0.982454
$\mathcal{H}_1$	0.064058	3.133343	0.982707
$\mathcal{H}_2$	0.064066	3.074796	0.982828
$\mathcal{H}_3$	0.064061	3.067171	0.982752
$\mathcal{H}_4$	0.064476	3.140062	0.989111
$\mathcal{H}_5$	0.063074	2.864557	0.967607
$\mathcal{H}_6$	0.062753	2.886719	0.962678
$\mathcal{H}_7$	0.064550	3.308343	0.990248
$\mathcal{H}_8$	0.052903	2.507125	0.811574
$\mathcal{H}_9$	0.058238	2.875078	0.893421
$\mathcal{H}_{10}$	0.047966	2.649886	0.735837
$\mathcal{H}_{11}$	0.062656	2.951492	0.961192
$\mathcal{H}_{12}$	0.061420	2.72449	0.942229
$\mathcal{H}_{13}$	0.059588	2.576688	0.914123
$\mathcal{H}_{14}$	0.054196	2.460642	0.831420
$\mathcal{H}_{15}$	0.043677	2.000000	0.670046
$\mathcal{H}_{16}$	0.048278	2.175527	0.740622
$\mathcal{H}_{17}$	0.064042	3.533776	0.982454

Appendix B: Proofs

Proof-1: We can prove Theorem 2 as:

(i) Set $\mathfrak{h} = 2$, then we have:

$T2NNTWAM\left(\tilde{\mathfrak{z}}_1, \ldots, \tilde{\mathfrak{z}}_{\mathfrak{h}}\right) = \oplus_{\mathfrak{g}=1}^{\mathfrak{h}=2} \mathfrak{s}_{\mathfrak{g}} \tilde{\mathfrak{z}}_{\mathfrak{g}} = \mathfrak{s}_1 \tilde{\mathfrak{z}}_1 \oplus \mathfrak{s}_2 \tilde{\mathfrak{z}}_2$

$$= \left\{\begin{array}{l}
\left(\begin{array}{l}
\frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_1\left(\tan\left(\frac{\pi\left(T_{T_{\mathfrak{z}_1}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_1\left(\tan\left(\frac{\pi\left(T_{I_{\mathfrak{z}_1}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_1\left(\tan\left(\frac{\pi\left(T_{F_{\mathfrak{z}_1}}(\hbar)\right)}{2}\right)\right)\right)\right)
\end{array}\right),\\
\left(\begin{array}{l}
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_1\left(\cot\left(\frac{\pi\left(I_{T_{\mathfrak{z}_1}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_1\left(\cot\left(\frac{\pi\left(I_{I_{\mathfrak{z}_1}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_1\left(\cot\left(\frac{\pi\left(I_{F_{\mathfrak{z}_1}}(\hbar)\right)}{2}\right)\right)\right)\right)
\end{array}\right),\\
\left(\begin{array}{l}
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_1\left(\cot\left(\frac{\pi\left(F_{T_{\mathfrak{z}_1}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_1\left(\cot\left(\frac{\pi\left(F_{I_{\mathfrak{z}_1}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_1\left(\cot\left(\frac{\pi\left(F_{F_{\mathfrak{z}_1}}(\hbar)\right)}{2}\right)\right)\right)\right)
\end{array}\right)
\end{array}\right\} \oplus \left\{\begin{array}{l}
\left(\begin{array}{l}
\frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_2\left(\tan\left(\frac{\pi\left(T_{T_{\mathfrak{z}_2}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_2\left(\tan\left(\frac{\pi\left(T_{I_{\mathfrak{z}_2}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_2\left(\tan\left(\frac{\pi\left(T_{F_{\mathfrak{z}_2}}(\hbar)\right)}{2}\right)\right)\right)\right)
\end{array}\right),\\
\left(\begin{array}{l}
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_2\left(\cot\left(\frac{\pi\left(I_{T_{\mathfrak{z}_2}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_2\left(\cot\left(\frac{\pi\left(I_{I_{\mathfrak{z}_2}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_2\left(\cot\left(\frac{\pi\left(I_{F_{\mathfrak{z}_2}}(\hbar)\right)}{2}\right)\right)\right)\right)
\end{array}\right),\\
\left(\begin{array}{l}
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_2\left(\cot\left(\frac{\pi\left(F_{T_{\mathfrak{z}_2}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_2\left(\cot\left(\frac{\pi\left(F_{I_{\mathfrak{z}_2}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_2\left(\cot\left(\frac{\pi\left(F_{F_{\mathfrak{z}_2}}(\hbar)\right)}{2}\right)\right)\right)\right)
\end{array}\right)
\end{array}\right\} = \left\{\begin{array}{l}
\left(\begin{array}{l}
\frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=2}\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(T_{T_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=2}\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(T_{I_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=2}\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(T_{F_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)
\end{array}\right),\\
\left(\begin{array}{l}
\frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=2}\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(I_{T_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=2}\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(I_{I_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=2}\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(I_{F_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)
\end{array}\right),\\
\left(\begin{array}{l}
\frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=2}\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(F_{T_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=2}\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(F_{I_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=2}\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(F_{F_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)
\end{array}\right)
\end{array}\right\}$$

(ii) Set $\mathfrak{h} = m$, then we have:

$$T2NNTWAM(\tilde{\mathfrak{z}}_1,\tilde{\mathfrak{z}}_2,\dots,\tilde{\mathfrak{z}}_{\mathfrak{h}}) = \oplus_{\mathfrak{g}=1}^{\mathfrak{h}=m} \mathfrak{s}_{\mathfrak{g}}\tilde{\mathfrak{z}}_{\mathfrak{g}} = \mathfrak{s}_1\tilde{\mathfrak{z}}_1 \oplus \dots \oplus \mathfrak{s}_m\tilde{\mathfrak{z}}_m$$

$$= \left\{\begin{array}{c}
\left(\begin{array}{c}
\frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_1\left(\tan\left(\frac{\pi\left(T_{T_{\mathfrak{z}_1}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_1\left(\tan\left(\frac{\pi\left(T_{I_{\mathfrak{z}_1}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_1\left(\tan\left(\frac{\pi\left(T_{F_{\mathfrak{z}_1}}(\hbar)\right)}{2}\right)\right)\right)\right)
\end{array}\right),\\
\left(\begin{array}{c}
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_1\left(\cot\left(\frac{\pi\left(I_{T_{\mathfrak{z}_1}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_1\left(\cot\left(\frac{\pi\left(I_{I_{\mathfrak{z}_1}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_1\left(\cot\left(\frac{\pi\left(I_{F_{\mathfrak{z}_1}}(\hbar)\right)}{2}\right)\right)\right)\right)
\end{array}\right),\\
\left(\begin{array}{c}
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_1\left(\cot\left(\frac{\pi\left(F_{T_{\mathfrak{z}_1}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_1\left(\cot\left(\frac{\pi\left(F_{I_{\mathfrak{z}_1}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_1\left(\cot\left(\frac{\pi\left(F_{F_{\mathfrak{z}_1}}(\hbar)\right)}{2}\right)\right)\right)\right)
\end{array}\right)
\end{array}\right\} \oplus \dots \oplus \left\{\begin{array}{c}
\left(\begin{array}{c}
\frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_m\left(\tan\left(\frac{\pi\left(T_{T_{\mathfrak{z}_m}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_m\left(\tan\left(\frac{\pi\left(T_{I_{\mathfrak{z}_m}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_m\left(\tan\left(\frac{\pi\left(T_{F_{\mathfrak{z}_m}}(\hbar)\right)}{2}\right)\right)\right)\right)
\end{array}\right),\\
\left(\begin{array}{c}
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_m\left(\cot\left(\frac{\pi\left(I_{T_{\mathfrak{z}_m}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_m\left(\cot\left(\frac{\pi\left(I_{I_{\mathfrak{z}_m}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_m\left(\cot\left(\frac{\pi\left(I_{F_{\mathfrak{z}_m}}(\hbar)\right)}{2}\right)\right)\right)\right)
\end{array}\right),\\
\left(\begin{array}{c}
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_m\left(\cot\left(\frac{\pi\left(F_{T_{\mathfrak{z}_m}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_m\left(\cot\left(\frac{\pi\left(F_{I_{\mathfrak{z}_m}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_m\left(\cot\left(\frac{\pi\left(F_{F_{\mathfrak{z}_m}}(\hbar)\right)}{2}\right)\right)\right)\right)
\end{array}\right)
\end{array}\right\} = \left\{\begin{array}{c}
\left(\begin{array}{c}
\frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(T_{T_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(T_{I_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(T_{F_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)
\end{array}\right),\\
\left(\begin{array}{c}
\frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(I_{T_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(I_{I_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(I_{F_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)
\end{array}\right),\\
\left(\begin{array}{c}
\frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(F_{T_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(F_{I_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(F_{F_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)
\end{array}\right)
\end{array}\right\}$$

(iii) Set $\boldsymbol{\mathfrak{h} = m + 1}$, then we have:

$$T2NNTWAM(\tilde{\mathfrak{z}}_1,\tilde{\mathfrak{z}}_2,\dots,\tilde{\mathfrak{z}}_{\mathfrak{h}}) = \oplus_{\mathfrak{g}=1}^{\mathfrak{h}=m+1} \mathfrak{s}_{\mathfrak{g}}\tilde{\mathfrak{z}}_{\mathfrak{g}} = \mathfrak{s}_1\tilde{\mathfrak{z}}_1 \oplus \dots \oplus \mathfrak{s}_m\tilde{\mathfrak{z}}_m \oplus \mathfrak{s}_{m+1}\tilde{\mathfrak{z}}_{m+1}$$

$$= \left\{\begin{array}{c}
\left(\begin{array}{c}
\frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(T_{T_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(T_{I_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(T_{F_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)
\end{array}\right),\\
\left(\begin{array}{c}
\frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(I_{T_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(I_{I_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(I_{F_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)
\end{array}\right),\\
\left(\begin{array}{c}
\frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(F_{T_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(F_{I_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(F_{F_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)
\end{array}\right)
\end{array}\right\} \oplus \left\{\begin{array}{c}
\left(\begin{array}{c}
\frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_{m+1}\left(\tan\left(\frac{\pi\left(T_{T_{\mathfrak{z}_{m+1}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_{m+1}\left(\tan\left(\frac{\pi\left(T_{I_{\mathfrak{z}_{m+1}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_{m+1}\left(\tan\left(\frac{\pi\left(T_{F_{\mathfrak{z}_{m+1}}}(\hbar)\right)}{2}\right)\right)\right)\right)
\end{array}\right),\\
\left(\begin{array}{c}
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_{m+1}\left(\cot\left(\frac{\pi\left(I_{T_{\mathfrak{z}_{m+1}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_{m+1}\left(\cot\left(\frac{\pi\left(I_{I_{\mathfrak{z}_{m+1}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_{m+1}\left(\cot\left(\frac{\pi\left(I_{F_{\mathfrak{z}_{m+1}}}(\hbar)\right)}{2}\right)\right)\right)\right)
\end{array}\right),\\
\left(\begin{array}{c}
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_{m+1}\left(\cot\left(\frac{\pi\left(F_{T_{\mathfrak{z}_{m+1}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_{m+1}\left(\cot\left(\frac{\pi\left(F_{I_{\mathfrak{z}_{m+1}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_{m+1}\left(\cot\left(\frac{\pi\left(F_{F_{\mathfrak{z}_{m+1}}}(\hbar)\right)}{2}\right)\right)\right)\right)
\end{array}\right)
\end{array}\right\} = \left\{\begin{array}{c}
\left(\begin{array}{c}
\frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m+1}\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(T_{T_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m+1}\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(T_{I_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m+1}\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(T_{F_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)
\end{array}\right),\\
\left(\begin{array}{c}
\frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m+1}\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(I_{T_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m+1}\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(I_{I_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m+1}\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(I_{F_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)
\end{array}\right),\\
\left(\begin{array}{c}
\frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m+1}\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(F_{T_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m+1}\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(F_{I_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right),\\
\frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m+1}\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(F_{F_{\mathfrak{z}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)
\end{array}\right)
\end{array}\right\}$$

Proof-2: We can prove *monotonicity* for Theorem 3 as:
Since $T_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar) \geq T_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}^*}(\hbar)$ and > 0 $and\ \pi = 3.14$, then:

$$\frac{\pi\left(T_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar)\right)}{2} \geq \frac{\pi\left(T_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}^*}(\hbar)\right)}{2}, \quad tan\left(\frac{\pi\left(T_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right) \geq tan\left(\frac{\pi\left(T_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}^*}(\hbar)\right)}{2}\right), \ \mathfrak{s}_{\mathfrak{g}}\left(tan\left(\frac{\pi\left(T_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right) \geq \mathfrak{s}_{\mathfrak{g}}\left(tan\left(\frac{\pi\left(T_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}^*}(\hbar)\right)}{2}\right)\right),$$

$$\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\left(\mathfrak{s}_{\mathfrak{g}}\left(tan\left(\frac{\pi\left(T_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right) \geq \sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\left(\mathfrak{s}_{\mathfrak{g}}\left(tan\left(\frac{\pi\left(T_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}^*}(\hbar)\right)}{2}\right)\right)\right),$$

$$tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\left(\mathfrak{s}_{\mathfrak{g}}\left(tan\left(\frac{\pi\left(T_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right) \geq tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\left(\mathfrak{s}_{\mathfrak{g}}\left(tan\left(\frac{\pi\left(T_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}^*}(\hbar)\right)}{2}\right)\right)\right)\right),$$

$$\frac{2}{\pi}\left(tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\left(\mathfrak{s}_{\mathfrak{g}}\left(tan\left(\frac{\pi\left(T_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right) \geq \frac{2}{\pi}\left(tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\left(\mathfrak{s}_{\mathfrak{g}}\left(tan\left(\frac{\pi\left(T_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}^*}(\hbar)\right)}{2}\right)\right)\right)\right)\right).$$

Similarly, we also have $T_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar) \geq T_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}^*}(\hbar), T_{I_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar) \geq T_{I_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}^*}(\hbar), T_{F_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar) \geq T_{F_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}^*}(\hbar)$, and $I_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar) \leq I_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}^*}(\hbar), I_{I_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar) \leq I_{I_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}^*}(\hbar), I_{F_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar) \leq I_{F_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}^*}(\hbar)$, and $F_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar) \geq F_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}^*}(\hbar), F_{I_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar) \leq F_{I_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}^*}(\hbar), F_{F_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar) \leq F_{F_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}^*}(\hbar)$, and $T2NNTWAM(\tilde{\mathfrak{z}}_1, \tilde{\mathfrak{z}}_2, \ldots, \tilde{\mathfrak{z}}_{\mathfrak{h}}) \geq T2NNTWAM(\tilde{\mathfrak{z}}_1^*, \tilde{\mathfrak{z}}_2^*, \ldots, \tilde{\mathfrak{z}}_{\mathfrak{h}}^*)$.

Proof-3: We can prove *idempotency* for Theorem 3 as:
Since $T_{T_{\tilde{\mathfrak{z}}}}(\hbar) = T_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar)$ and $\mathfrak{s} > 0$ $and\ \pi = 3.14$, then:

$$\frac{\pi\left(T_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar)\right)}{2} = \frac{\pi\left(T_{T_{\tilde{\mathfrak{z}}}}(\hbar)\right)}{2}, tan\left(\frac{\pi\left(T_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right) = tan\left(\frac{\pi\left(T_{T_{\tilde{\mathfrak{z}}}}(\hbar)\right)}{2}\right), \mathfrak{s}_{\mathfrak{g}}\left(tan\left(\frac{\pi\left(T_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right) = \mathfrak{s}\left(tan\left(\frac{\pi\left(T_{T_{\tilde{\mathfrak{z}}}}(\hbar)\right)}{2}\right)\right),$$

$$\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\left(\mathfrak{s}_{\mathfrak{g}}\left(tan\left(\frac{\pi\left(T_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right) = \mathfrak{s}\left(tan\left(\frac{\pi\left(T_{T_{\tilde{\mathfrak{z}}}}(\hbar)\right)}{2}\right)\right),$$

$$tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\left(\mathfrak{s}_{\mathfrak{g}}\left(tan\left(\frac{\pi\left(T_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right) = tan^{-1}\left(\mathfrak{s}\left(tan\left(\frac{\pi\left(T_{T_{\tilde{\mathfrak{z}}}}(\hbar)\right)}{2}\right)\right)\right),$$

$$\frac{2}{\pi}\left(tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\left(\mathfrak{s}_{\mathfrak{g}}\left(tan\left(\frac{\pi\left(T_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right) = \frac{2}{\pi}\left(tan^{-1}\left(\mathfrak{s}\left(tan\left(\frac{\pi\left(T_{T_{\tilde{\mathfrak{z}}}}(\hbar)\right)}{2}\right)\right)\right)\right).$$

Similarly, we also have $T_{T_{\tilde{\mathfrak{z}}}}(\hbar) = T_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar), T_{I_{\tilde{\mathfrak{z}}}}(\hbar) = T_{I_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar),\ T_{F_{\tilde{\mathfrak{z}}}}(\hbar) = T_{F_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar)$, and $I_{T_{\tilde{\mathfrak{z}}}}(\hbar) = I_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar), I_{I_{\tilde{\mathfrak{z}}}}(\hbar) = I_{I_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar),\ I_{F_{\tilde{\mathfrak{z}}}}(\hbar) = I_{F_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar)$, and $F_{T_{\tilde{\mathfrak{z}}}}(\hbar) = F_{T_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar), F_{I_{\tilde{\mathfrak{z}}}}(\hbar) = F_{I_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar),\ F_{F_{\tilde{\mathfrak{z}}}}(\hbar) = F_{F_{\tilde{\mathfrak{z}}_{\mathfrak{g}}}}(\hbar)$, and $T2NNTWAM(\tilde{\mathfrak{z}}_1, \tilde{\mathfrak{z}}_2, \ldots, \tilde{\mathfrak{z}}_{\mathfrak{h}}) = \tilde{\mathfrak{z}}$.

Proof-4: We can prove Theorem 4 as:

(i) Set $\mathfrak{h}=\mathbf{2}$, then we have:

$$T2NNTWGM\left(\tilde{Ʒ}_1,\dots,\tilde{Ʒ}_{\mathfrak{h}}\right)=\otimes_{\mathfrak{g}=1}^{\mathfrak{h}=2}\left(\tilde{Ʒ}_{\mathfrak{g}}\right)^{\mathfrak{s}_{\mathfrak{g}}}=\left(\tilde{Ʒ}_1\right)^{\mathfrak{s}_1}\otimes\left(\tilde{Ʒ}_2\right)^{\mathfrak{s}_2}$$

$$=\left\{\begin{matrix}\begin{pmatrix}\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_1\left(\cot\left(\frac{\pi\left(T_{T_{Ʒ_1}}(\hbar)\right)}{2}\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_1\left(\cot\left(\frac{\pi\left(T_{I_{Ʒ_1}}(\hbar)\right)}{2}\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_1\left(\cot\left(\frac{\pi\left(T_{F_{Ʒ_1}}(\hbar)\right)}{2}\right)\right)\right)\right)\end{pmatrix},\\ \begin{pmatrix}\frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_1\left(\tan\left(\frac{\pi\left(I_{T_{Ʒ_1}}(\hbar)\right)}{2}\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_1\left(\tan\left(\frac{\pi\left(I_{I_{Ʒ_1}}(\hbar)\right)}{2}\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_1\left(\tan\left(\frac{\pi\left(I_{F_{Ʒ_1}}(\hbar)\right)}{2}\right)\right)\right)\right)\end{pmatrix},\\ \begin{pmatrix}\frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_1\left(\tan\left(\frac{\pi\left(F_{T_{Ʒ_1}}(\hbar)\right)}{2}\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_1\left(\tan\left(\frac{\pi\left(F_{I_{Ʒ_1}}(\hbar)\right)}{2}\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_1\left(\tan\left(\frac{\pi\left(F_{F_{Ʒ_1}}(\hbar)\right)}{2}\right)\right)\right)\right)\end{pmatrix}\end{matrix}\right\}\oplus\left\{\begin{matrix}\begin{pmatrix}\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_2\left(\cot\left(\frac{\pi\left(T_{T_{Ʒ_2}}(\hbar)\right)}{2}\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_2\left(\cot\left(\frac{\pi\left(T_{I_{Ʒ_2}}(\hbar)\right)}{2}\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_2\left(\cot\left(\frac{\pi\left(T_{F_{Ʒ_2}}(\hbar)\right)}{2}\right)\right)\right)\right)\end{pmatrix},\\ \begin{pmatrix}\frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_2\left(\tan\left(\frac{\pi\left(I_{T_{Ʒ_2}}(\hbar)\right)}{2}\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_2\left(\tan\left(\frac{\pi\left(I_{I_{Ʒ_2}}(\hbar)\right)}{2}\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_2\left(\tan\left(\frac{\pi\left(I_{F_{Ʒ_2}}(\hbar)\right)}{2}\right)\right)\right)\right)\end{pmatrix},\\ \begin{pmatrix}\frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_2\left(\tan\left(\frac{\pi\left(F_{T_{Ʒ_2}}(\hbar)\right)}{2}\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_2\left(\tan\left(\frac{\pi\left(F_{I_{Ʒ_2}}(\hbar)\right)}{2}\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_2\left(\tan\left(\frac{\pi\left(F_{F_{Ʒ_2}}(\hbar)\right)}{2}\right)\right)\right)\right)\end{pmatrix}\end{matrix}\right\}=\left\{\begin{matrix}\begin{pmatrix}\frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=2}\left(\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(T_{T_{Ʒ_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=2}\left(\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(T_{I_{Ʒ_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=2}\left(\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(T_{F_{Ʒ_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right)\end{pmatrix},\\ \begin{pmatrix}\frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=2}\left(\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(I_{T_{Ʒ_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=2}\left(\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(I_{I_{Ʒ_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=2}\left(\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(I_{F_{Ʒ_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right)\end{pmatrix},\\ \begin{pmatrix}\frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=2}\left(\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(F_{T_{Ʒ_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=2}\left(\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(F_{I_{Ʒ_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=2}\left(\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(F_{F_{Ʒ_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right)\end{pmatrix}\end{matrix}\right\}$$

(ii) Set $\mathfrak{h}=\boldsymbol{m}$, then we have:

$$T2NNTWGM\left(\tilde{Ʒ}_1,\tilde{Ʒ}_2,\dots,\tilde{Ʒ}_{\mathfrak{h}}\right)=\otimes_{\mathfrak{g}=1}^{\mathfrak{h}=m}\left(\tilde{Ʒ}_{\mathfrak{g}}\right)^{\mathfrak{s}_{\mathfrak{g}}}=\left(\tilde{Ʒ}_1\right)^{\mathfrak{s}_1}\otimes\dots\otimes\left(\tilde{Ʒ}_m\right)^{\mathfrak{s}_m}$$

$$=\left\{\begin{matrix}\begin{pmatrix}\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_1\left(\cot\left(\frac{\pi\left(T_{T_{Ʒ_1}}(\hbar)\right)}{2}\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_1\left(\cot\left(\frac{\pi\left(T_{I_{Ʒ_1}}(\hbar)\right)}{2}\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_1\left(\cot\left(\frac{\pi\left(T_{F_{Ʒ_1}}(\hbar)\right)}{2}\right)\right)\right)\right)\end{pmatrix},\\ \begin{pmatrix}\frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_1\left(\tan\left(\frac{\pi\left(I_{T_{Ʒ_1}}(\hbar)\right)}{2}\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_1\left(\tan\left(\frac{\pi\left(I_{I_{Ʒ_1}}(\hbar)\right)}{2}\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_1\left(\tan\left(\frac{\pi\left(I_{F_{Ʒ_1}}(\hbar)\right)}{2}\right)\right)\right)\right)\end{pmatrix},\\ \begin{pmatrix}\frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_1\left(\tan\left(\frac{\pi\left(F_{T_{Ʒ_1}}(\hbar)\right)}{2}\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_1\left(\tan\left(\frac{\pi\left(F_{I_{Ʒ_1}}(\hbar)\right)}{2}\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_1\left(\tan\left(\frac{\pi\left(F_{F_{Ʒ_1}}(\hbar)\right)}{2}\right)\right)\right)\right)\end{pmatrix}\end{matrix}\right\}\oplus\dots\oplus\left\{\begin{matrix}\begin{pmatrix}\frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_m\left(\cot\left(\frac{\pi\left(T_{T_{Ʒ_m}}(\hbar)\right)}{2}\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_m\left(\cot\left(\frac{\pi\left(T_{I_{Ʒ_m}}(\hbar)\right)}{2}\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_m\left(\cot\left(\frac{\pi\left(T_{F_{Ʒ_m}}(\hbar)\right)}{2}\right)\right)\right)\right)\end{pmatrix},\\ \begin{pmatrix}\frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_m\left(\tan\left(\frac{\pi\left(I_{T_{Ʒ_m}}(\hbar)\right)}{2}\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_m\left(\tan\left(\frac{\pi\left(I_{I_{Ʒ_m}}(\hbar)\right)}{2}\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_m\left(\tan\left(\frac{\pi\left(I_{F_{Ʒ_m}}(\hbar)\right)}{2}\right)\right)\right)\right)\end{pmatrix},\\ \begin{pmatrix}\frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_m\left(\tan\left(\frac{\pi\left(F_{T_{Ʒ_m}}(\hbar)\right)}{2}\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_m\left(\tan\left(\frac{\pi\left(F_{I_{Ʒ_m}}(\hbar)\right)}{2}\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_m\left(\tan\left(\frac{\pi\left(F_{F_{Ʒ_m}}(\hbar)\right)}{2}\right)\right)\right)\right)\end{pmatrix}\end{matrix}\right\}=\left\{\begin{matrix}\begin{pmatrix}\frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\left(\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(T_{T_{Ʒ_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\left(\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(T_{I_{Ʒ_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\left(\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(T_{F_{Ʒ_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right)\end{pmatrix},\\ \begin{pmatrix}\frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\left(\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(I_{T_{Ʒ_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\left(\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(I_{I_{Ʒ_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\left(\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(I_{F_{Ʒ_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right)\end{pmatrix},\\ \begin{pmatrix}\frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\left(\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(F_{T_{Ʒ_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\left(\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(F_{I_{Ʒ_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right),\\ \frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\left(\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(F_{F_{Ʒ_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right)\end{pmatrix}\end{matrix}\right\}$$

(iii) Set $\mathfrak{h}=\boldsymbol{m}+\mathbf{1}$, then we have:

$$T2NNTWGM\left(\tilde{Ʒ}_1,\tilde{Ʒ}_2,\dots,\tilde{Ʒ}_{\mathfrak{h}}\right)=\otimes_{\mathfrak{g}=1}^{\mathfrak{h}=m+1}\left(\tilde{Ʒ}_{\mathfrak{g}}\right)^{\mathfrak{s}_{\mathfrak{g}}}=\left(\tilde{Ʒ}_1\right)^{\mathfrak{s}_1}\otimes\dots\otimes\left(\tilde{Ʒ}_m\right)^{\mathfrak{s}_m}\otimes\left(\tilde{Ʒ}_{m+1}\right)^{\mathfrak{s}_{m+1}}$$

$$
= \left\{ \begin{pmatrix} \begin{pmatrix} \frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\left(\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(T_{T_{3_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right), \\ \frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\left(\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(T_{I_{3_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right), \\ \frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\left(\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(T_{F_{3_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right) \end{pmatrix}, \\ \begin{pmatrix} \frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\left(\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(I_{T_{3_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right), \\ \frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\left(\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(I_{I_{3_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right), \\ \frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\left(\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(I_{F_{3_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right) \end{pmatrix}, \\ \begin{pmatrix} \frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\left(\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(F_{T_{3_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right), \\ \frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\left(\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(F_{I_{3_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right), \\ \frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m}\left(\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(F_{F_{3_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right) \end{pmatrix} \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} \begin{pmatrix} \frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_{m+1}\left(\cot\left(\frac{\pi\left(T_{T_{3_{m+1}}}(\hbar)\right)}{2}\right)\right)\right)\right), \\ \frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_{m+1}\left(\cot\left(\frac{\pi\left(T_{I_{3_{m+1}}}(\hbar)\right)}{2}\right)\right)\right)\right), \\ \frac{2}{\pi}\left(\cot^{-1}\left(\mathfrak{s}_{m+1}\left(\cot\left(\frac{\pi\left(T_{F_{3_{m+1}}}(\hbar)\right)}{2}\right)\right)\right)\right) \end{pmatrix}, \\ \begin{pmatrix} \frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_{m+1}\left(\tan\left(\frac{\pi\left(I_{T_{3_{m+1}}}(\hbar)\right)}{2}\right)\right)\right)\right), \\ \frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_{m+1}\left(\tan\left(\frac{\pi\left(I_{I_{3_{m+1}}}(\hbar)\right)}{2}\right)\right)\right)\right), \\ \frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_{m+1}\left(\tan\left(\frac{\pi\left(I_{F_{3_{m+1}}}(\hbar)\right)}{2}\right)\right)\right)\right) \end{pmatrix}, \\ \begin{pmatrix} \frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_{m+1}\left(\tan\left(\frac{\pi\left(F_{T_{3_{m+1}}}(\hbar)\right)}{2}\right)\right)\right)\right), \\ \frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_{m+1}\left(\tan\left(\frac{\pi\left(F_{I_{3_{m+1}}}(\hbar)\right)}{2}\right)\right)\right)\right), \\ \frac{2}{\pi}\left(\tan^{-1}\left(\mathfrak{s}_{m+1}\left(\tan\left(\frac{\pi\left(F_{F_{3_{m+1}}}(\hbar)\right)}{2}\right)\right)\right)\right) \end{pmatrix} \end{pmatrix} \right\} = \left\{ \begin{pmatrix} \begin{pmatrix} \frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m+1}\left(\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(T_{T_{3_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right), \\ \frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m+1}\left(\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(T_{I_{3_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right), \\ \frac{2}{\pi}\left(\cot^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m+1}\left(\mathfrak{s}_{\mathfrak{g}}\left(\cot\left(\frac{\pi\left(T_{F_{3_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right) \end{pmatrix}, \\ \begin{pmatrix} \frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m+1}\left(\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(I_{T_{3_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right), \\ \frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m+1}\left(\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(I_{I_{3_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right), \\ \frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m+1}\left(\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(I_{F_{3_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right) \end{pmatrix}, \\ \begin{pmatrix} \frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m+1}\left(\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(F_{T_{3_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right), \\ \frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m+1}\left(\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(F_{I_{3_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right), \\ \frac{2}{\pi}\left(\tan^{-1}\left(\sum_{\mathfrak{g}=1}^{\mathfrak{h}=m+1}\left(\mathfrak{s}_{\mathfrak{g}}\left(\tan\left(\frac{\pi\left(F_{F_{3_{\mathfrak{g}}}}(\hbar)\right)}{2}\right)\right)\right)\right)\right) \end{pmatrix} \end{pmatrix} \right\}
$$

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