SCIENTIFIC OASIS

Spectrum of Engineering and Management Sciences

Journal homepage: www.sems-journal.org
ISSN: 3009-3309

Enhancing Artificial Intelligence Models with Interval-Valued Picture Fuzzy Sets and Sugeno-Weber Triangular Norms

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ARTICLE INFO

Article history:
Received 3 January 2025
Received in revised form 10 March 2025
Accepted 14 March 2025
Available online 14 March 2025

Keywords:
Picture Fuzzy Sets; Aggregation Operators; Sugeno-Weber Triangular Norms; Intelligent Selection Process; Artificial Intelligence; Management; Multi-Attribute Decision-Making

ABSTRACT

This work aims to improve intelligence decision-making using interval-valued picture fuzzy sets (IVPFS). In particular, it explores using Sugeno-Weber (SW) norms in IVPFS data recording, providing reliable estimates important for decision-making. This paper introduces a new class of aggregation operators such as the interval-valued picture fuzzy Sugeno-Weber power average (IVPFSWPA), interval-valued picture fuzzy Sugeno-Weber power geometric (IVPFSWPG), interval-valued picture fuzzy Sugeno-Weber power weighted average (IVPFSWPWA), and interval-valued picture fuzzy Sugeno-Weber power weighted geometric (IVPFSWPWG) operators. The real-life characteristics and specific situations of these operators are described as well as how they adapt to real-life situations. The new multi-attribute decision-making method suitable for many practical applications with different requirements or functions is proposed. An example of an intelligent selection process is given to demonstrate its effectiveness. In addition, a general comparative method is proposed to demonstrate the effectiveness and suitability of the collective strategy by comparing its results with existing methods. The study concludes by summarizing its findings and discussing its prospects, highlighting the potential contribution of the proposed studies to the advancement of cutting-edge technology in a dynamic and complex environment.

1. Introduction

The main objective of a multi-attribute decision-making (MADM) problem is to evaluate or define the consensus that leads to the minimum choice, given the preference values given by experts or people making decisions based on their values [1,2]. Experts consider it strange and difficult to give a number to all needs in real-world problems due to the uncertainty and complexity of the decision.

In Zadeh [3] fuzzy set (FS) theory, the contribution of an element to the problem is defined by its membership grade (MG). Sometimes the applicability of the FS strategy turns out to be limited. For

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https://doi.org/10.31181/sems31202540g

example, all data for two or more variables are different from each other. With the help of membership grade (MG) and non-membership grade (NMG), Atanassov [4] obtained the intuitionistic fuzzy system (IFS), which is a modification of FS, that can be followed because the information is difficult and uncertain. IFS theory is useful in many problems [5]. However, the IFS dataset is limited and relies on the arbitrary requirement that the MG and NMG numbers be in cell $[0, 1]$. Yager [6] proposed the concept of Pythagorean fuzzy sets (PyFS), a modified form of IFS, to deal with imprecise and negative data. The requirement that the MG and NMG numbers fall within the $[0,1]$ distribution is maintained and enforced by PyFS, just like by IFS. PyFS cannot follow MG and NMG if the sum of the squares of MG and NMG is greater than the unit average; for instance, if MG is set to 0.5 and NMG is set to 0.9 [7]. In order to address this issue, Yager [8] suggested q-rung orthopair fuzzy sets (q-ROFS), which are capable of carrying out the aforementioned difficult and unclear tasks. A pair of MGS and NMG for which the total of specific q powers must fall within the unit period is known as the q-ROF value (q-ROFV). Using the q parameter, we can select MG and NMG from $[0, 1]$ for each pair (MG, NMG) that satisfies $0 \leq MG^q + NMG^q \leq 1$. Cuong [9] proposed a picture fuzzy set (PFS). The PFS range also includes the $[0, 1]$ range [10].

Sarfraz *et al.* [11] conducted a comparative study based on the concept of pre-assembled workers based on IFS. The idea of Aczel-Alsina prioritization using complex IFS was proposed by Ullah *et al.* [12]. Nonlinear optimization and Sugeno-Weber's triangular norms (TNs) idea were presented by Ghodousian *et al.* [13]. The Sugeno-Weber TN theory of FS was developed by Kauers *et al.* [14]. Sarkar *et al.* [15] offered some useful mathematical techniques for handling uncertain data from a human standpoint by utilizing the Sugeno-Weber *t*-norm and *t*-conorm characteristics. The comparison of *t*-standard deviations in classification issues is explained in [16]. Sugeno-Weber and parametric statistical analysis were contrasted by Troiano *et al.* [17]. The Sugeno-Weber idea was put forth by Klement *et al.* [18] utilizing the TN family. The use of Sugeno-Weber fixed point theory in probability measurement is demonstrated by Hadzic *et al.* [19].

The main contribution of this article is as follows:

i. Based on our different views, we present some powerful mathematical ideas, including the interval-valued picture fuzzy Sugeno-Weber power average (IVPFSWPA), interval-valued picture fuzzy Sugeno-Weber power geometric (IVPFSWPG), interval-valued picture fuzzy Sugeno-Weber power weighted average (IVPFSWPWA), and interval-valued picture fuzzy Sugeno-Weber power weighted geometric (IVPFSWPWG) operators, along with important properties and special cases.
ii. With the help of the ideas presented, we evaluated the suitability and effectiveness of the established aggregation operators (AOs) and developed a complex decision-making process.
iii. Using the design, the computational paradigm is also examined to assess the plausible reliability of intelligence in certain important aspects and properties.

In Section 1, we review some recent literature on interval-valued picture fuzzy sets (IVPFS). Section 2 contains an overview of the Sugeno-Weber *t*-norm (SWTN) of the IVPFS. Some of the SWTN operating requirements of IVPFS are included in Section 3. In Section 4, we introduce the new AO for MADM's IVPFS operator. In Section 5, we present a numerical example. Chapter 6 gives conclusions of the work.

2. Preliminaries

In order to create and validate the suggested research topic, we look at the basic ideas of Sugeno-Weber triangular norms, or PFSs, and the basic laws that govern them.

Definition 1. The following gives a PFS α of the discourse universe ω:

$$\boldsymbol{\alpha} = \{\langle \boldsymbol{\tau}, (\boldsymbol{\mu}(\boldsymbol{\tau}), \boldsymbol{\nu}(\boldsymbol{\tau}), \boldsymbol{\pi}(\boldsymbol{\tau})) | \boldsymbol{\tau} \in \boldsymbol{\omega}\rangle\}, \tag{1}$$

where $\mu(\tau): \omega \to [0,1]$, $\nu(\tau): \omega \to [0,1]$, and $\pi(\tau): \omega \to [0,1]$ specifies the levels of MD, NMD, and hesitancy, respectively, with:

$$\mathbf{0} \leq \boldsymbol{\mu}(\boldsymbol{\tau}) + \boldsymbol{\pi}(\boldsymbol{\tau}) + \boldsymbol{v}(\boldsymbol{\tau}) \leq \mathbf{1}, \forall \boldsymbol{\tau} \in \boldsymbol{\omega}. \tag{2}$$

Definition 2. For any PFS $\alpha = (\mu_\alpha(\tau), \nu_\alpha(\tau), \pi_\alpha(\tau))$ and $\beta = \left(\mu_\beta(\tau), \nu_\beta(\tau), \pi_\beta(\tau)\right)$, $\lambda_1, \lambda_2, \lambda_3 > 0$, a few essential Sugeno-Weber tool actions are stated:

$$\boldsymbol{\alpha} \cup \boldsymbol{\beta} = \left\{ \begin{array}{c} \boldsymbol{max}\{\boldsymbol{\mu}_\alpha(\boldsymbol{\tau}), \boldsymbol{\mu}_\beta(\boldsymbol{\tau})\}, \boldsymbol{min}\{\boldsymbol{v}_\alpha(\boldsymbol{\tau}), \boldsymbol{v}_\beta(\boldsymbol{\tau})\}, \\ \boldsymbol{min}\left\{ \begin{array}{c} \mathbf{1} - \left((\boldsymbol{max}\{\boldsymbol{\mu}_\alpha(\boldsymbol{\tau}), \boldsymbol{\mu}_\beta(\boldsymbol{\tau})\}) + (\boldsymbol{min}\{\boldsymbol{v}_\alpha(\boldsymbol{\tau}), \boldsymbol{v}_\beta(\boldsymbol{\tau})\})\right), \\ \boldsymbol{max}\{\boldsymbol{\pi}_\alpha(\boldsymbol{\tau}), \boldsymbol{\pi}_\beta(\boldsymbol{\tau})\} \end{array} \right\} \end{array} \right\}, \tag{3}$$

$$\boldsymbol{\alpha} \cap \boldsymbol{\beta} = \left\{ \begin{array}{c} \boldsymbol{min}\{\boldsymbol{\mu}_\alpha(\boldsymbol{\tau}), \boldsymbol{\mu}_\beta(\boldsymbol{\tau})\}, \boldsymbol{max}\{\boldsymbol{v}_\alpha(\boldsymbol{\tau}), \boldsymbol{v}_\beta(\boldsymbol{\tau})\}, \\ \boldsymbol{min}\left\{ \begin{array}{c} \mathbf{1} - \left((\boldsymbol{min}\{\boldsymbol{\mu}_\alpha(\boldsymbol{\tau}), \boldsymbol{\mu}_\beta(\boldsymbol{\tau})\}) + (\boldsymbol{max}\{\boldsymbol{v}_\alpha(\boldsymbol{\tau}), \boldsymbol{v}_\beta(\boldsymbol{\tau})\})\right), \\ \boldsymbol{min}\{\boldsymbol{\pi}_\alpha(\boldsymbol{\tau}), \boldsymbol{\pi}_\beta(\boldsymbol{\tau})\} \end{array} \right\} \end{array} \right\}, \tag{4}$$

$$\boldsymbol{\alpha} \oplus \boldsymbol{\beta} = \left\{ \begin{array}{c} \left(\boldsymbol{\mu}_\alpha(\boldsymbol{\tau}) + \boldsymbol{\mu}_\beta(\boldsymbol{\tau}) - \boldsymbol{\mu}_\alpha(\boldsymbol{\tau})\boldsymbol{\mu}_\beta(\boldsymbol{\tau})\right)^{\frac{1}{2}}, \\ \boldsymbol{v}_\alpha(\boldsymbol{\tau}), \boldsymbol{v}_\beta(\boldsymbol{\tau}) \left(\left(\mathbf{1} - \boldsymbol{\mu}_\beta(\boldsymbol{\tau})\right)\boldsymbol{\pi}_\alpha(\boldsymbol{\tau}) + (\mathbf{1} - \boldsymbol{\mu}_\alpha(\boldsymbol{\tau}))\boldsymbol{\pi}_\beta(\boldsymbol{\tau}) - \boldsymbol{\pi}_\alpha(\boldsymbol{\tau})\boldsymbol{\pi}_\beta(\boldsymbol{\tau})\right)^{\frac{1}{2}} \end{array} \right\}, \tag{5}$$

$$\boldsymbol{\alpha} \otimes \boldsymbol{\beta} = \left\{ \begin{array}{c} \boldsymbol{\mu}_\alpha(\boldsymbol{\tau}), \boldsymbol{\mu}_\beta(\boldsymbol{\tau}) \left(\boldsymbol{v}_\alpha(\boldsymbol{\tau}) + \boldsymbol{v}_\beta(\boldsymbol{\tau}) - \boldsymbol{v}_\alpha(\boldsymbol{\tau})\boldsymbol{v}_\beta(\boldsymbol{\tau})\right)^{\frac{1}{2}}, \\ \left(\left(\mathbf{1} - \boldsymbol{v}_\beta(\boldsymbol{\tau})\right)\boldsymbol{\pi}_\alpha(\boldsymbol{\tau}) + (\mathbf{1} - \boldsymbol{\mu}_\alpha(\boldsymbol{\tau}))\boldsymbol{\pi}_\beta(\boldsymbol{\tau}) - \boldsymbol{\pi}_\alpha(\boldsymbol{\tau})\boldsymbol{\pi}_\beta(\boldsymbol{\tau})\right)^{\frac{1}{2}} \end{array} \right\}, \tag{6}$$

$$\boldsymbol{\lambda\alpha} = \left\{\left(\mathbf{1} - (\mathbf{1} - \boldsymbol{\mu}_\alpha(\boldsymbol{\tau}))^{\boldsymbol{\lambda}}\right)^{\frac{1}{2}} . \left((\mathbf{1} - \boldsymbol{v}_\alpha(\boldsymbol{\tau}))^{\boldsymbol{\lambda}} - (\mathbf{1} - \boldsymbol{v}_\alpha(\boldsymbol{\tau}) - \boldsymbol{\pi}_\alpha(\boldsymbol{\tau}))^{\boldsymbol{\lambda}}\right)^{\frac{1}{2}}\right\}, \tag{7}$$

$$\boldsymbol{\alpha}^{\boldsymbol{\lambda}} = \left\{\boldsymbol{\mu}_\alpha^{\boldsymbol{\lambda}}(\boldsymbol{\tau}), \left(\mathbf{1} - (\mathbf{1} - \boldsymbol{v}_\alpha(\boldsymbol{\tau}))^{\boldsymbol{\lambda}}\right)^{\frac{1}{2}} . \left((\mathbf{1} - \boldsymbol{v}_\alpha(\boldsymbol{\tau}))^{\boldsymbol{\lambda}} - (\mathbf{1} - \boldsymbol{v}_\alpha(\boldsymbol{\tau}) - \boldsymbol{\pi}_\alpha(\boldsymbol{\tau}))^{\boldsymbol{\lambda}}\right)^{\frac{1}{2}}\right\}. \tag{8}$$

Definition 3: For PFSs $\alpha = (\mu_\alpha(\tau), \nu_\alpha(\tau), \pi_\alpha(\tau))$ and $\beta = \left(\mu_\beta(\tau), \nu_\beta(\tau), \pi_\beta(\tau)\right)$ the following is true under $\lambda_1, \lambda_2, \lambda_3 > 0$:

i. $\boldsymbol{\alpha} \oplus \boldsymbol{\beta} = \boldsymbol{\beta} \oplus \boldsymbol{\alpha}$,
ii. $\boldsymbol{\alpha} \otimes \boldsymbol{\beta} = \boldsymbol{\beta} \otimes \boldsymbol{\alpha}$,

iii. $\boldsymbol{\lambda(\alpha \oplus \beta) = \lambda\alpha \oplus \lambda\beta}$,
iv. $\boldsymbol{\lambda_1\alpha \oplus \lambda_2\alpha = (\lambda_1 \oplus \lambda_2)\alpha}$,
v. $\boldsymbol{(\alpha \otimes \beta)^\lambda = \alpha^\lambda \otimes \beta^\lambda}$,
vi. $\boldsymbol{\alpha^{\lambda_1} \otimes \alpha^{\lambda_2} = \alpha^{\lambda_1+\lambda_2}}$.

Definition 4. An interval-valued PFS of the discourse universe ω is $\alpha = \{\tau, ([\mu^l(\tau), \mu^u(\tau)], [v^l(\tau), v^u(\tau)], [\pi^l(\tau), \pi^u(\tau)])\}$ with $0 \leq (\mu)^u(\tau) + (v)^u(\tau) + (\pi)^u(\tau) \leq 1$ and $0 \leq v^l(\tau) \leq 1$. Here, $\mu(\tau) = [\mu^l(\tau), \mu^u(\tau)]$, $v(\tau) = [v^l(\tau), v^u(\tau)]$, and $\pi(\tau) = [\pi^l(\tau), \pi^u(\tau)]$. The expression $r(\tau) = \left[\sqrt{1 - (\mu)^u(\tau) + (v)^u(\tau) + (\pi)^u(\tau)}, \sqrt{1 - (\mu)^l(\tau) + ((v)^l)(\tau) + (\pi)^l(\tau)}\right]$ is termed as the DR. We call $\alpha = (\mu, v, \pi) = ([\mu^l(\tau), \mu^u(\tau)], [v^l(\tau), v^u(\tau)], [\pi^l(\tau), \pi^u(\tau)])$ as an IVSF number.

Definition 5. Let α, α_1, and α_2 be three IVSF sets. Then:

$$\boldsymbol{\alpha_1 \cup \alpha_2} = \begin{Bmatrix} \left[\boldsymbol{max\{\mu_1^l(\tau), \mu_2^l(\tau)\}, max\{\mu_1^u(\tau), \mu_2^u(\tau)\}}\right], \\ \left[\boldsymbol{min\{v_1^l(\tau), v_2^l(\tau)\}, min\{v_1^u(\tau), v_2^u(\tau)\}}\right], \\ \left[\boldsymbol{min\{\pi_1^l(\tau), \pi_2^l(\tau)\}, min\{\pi_1^U(\tau), \pi_2^U(\tau)\}}\right] \end{Bmatrix}, \tag{9}$$

$$\boldsymbol{\alpha_1 \cap \alpha_2} = \begin{Bmatrix} \left[\boldsymbol{min\{\mu_1^l(\tau), \mu_2^l(\tau)\}, min\{\mu_1^u(\tau), \mu_2^u(\tau)\}}\right], \\ \left[\boldsymbol{max\{v_1^l(\tau), v_2^l(\tau)\}, max\{v_1^u(\tau), v_2^u(\tau)\}}\right], \\ \left[\boldsymbol{min\{\pi_1^l(\tau), \pi_2^l(\tau)\}, min\{\pi_1^u(\tau), \pi_2^u(\tau)\}}\right] \end{Bmatrix}, \tag{10}$$

$$\boldsymbol{\alpha_1 \oplus \alpha_2} = \begin{pmatrix} \left[\sqrt{\left(\boldsymbol{\mu_1^l(\tau)} + \left(\boldsymbol{\mu_2^l(\tau)}\right) - \left(\boldsymbol{\mu_1^l(\tau)}\right).\left(\boldsymbol{\mu_2^l(\tau)}\right)\right)}, \sqrt{\left(\boldsymbol{\mu_1^u(\tau)}\right) + \left(\boldsymbol{\mu_2^u(\tau)}\right) - \left(\boldsymbol{\mu_1^u(\tau)}\right).\left(\boldsymbol{\mu_2^u(\tau)}\right)}\right], \\ \left[\boldsymbol{v_1^l(\tau)v_2^l(\tau), v_1^u(\tau)v_2^u(\tau)}\right], \\ \left[\boldsymbol{\pi_1^l(\tau)\pi_2^l(\tau), \pi_1^u(\tau)\pi_2^u(\tau)}\right] \end{pmatrix}, \tag{11}$$

$$\boldsymbol{\alpha_1 \otimes \alpha_2} = \begin{pmatrix} \left[\boldsymbol{\mu_1^l(\tau)\mu_2^l(\tau), \mu_1^U(\tau)\mu_2^U(\tau)}\right], \\ \left[\sqrt{\left(\left(\boldsymbol{v_1^l(\tau)}\right) + \left(\boldsymbol{v_2^l(\tau)}\right) - \left(\boldsymbol{v_1^l(\tau)}\right).\left(\boldsymbol{v_2^l(\tau)}\right)\right)}, \sqrt{\left(\boldsymbol{v_1^u(\tau)}\right) + \left(\boldsymbol{v_2^u(\tau)}\right) - \left(\boldsymbol{v_1^u(\tau)}\right).\left(\boldsymbol{v_2^u(\tau)}\right)}\right], \\ \left[\sqrt{\left(\left(\boldsymbol{\pi_1^l(\tau)}\right) + \left(\boldsymbol{\pi_2^l(\tau)}\right) - \left(\boldsymbol{\pi_1^l(\tau)}\right).\left(\boldsymbol{\pi_2^l(\tau)}\right)\right)}, \sqrt{\left(\boldsymbol{\pi_1^u(\tau)}\right) + \left(\boldsymbol{\pi_2^u(\tau)}\right) - \left(\boldsymbol{\pi_1^u(\tau)}\right).\left(\boldsymbol{\pi_2^u(\tau)}\right)}\right] \end{pmatrix}, \tag{12}$$

$$\boldsymbol{\lambda\alpha} = \left(\sqrt{\boldsymbol{1} - \left(\boldsymbol{1} - \left(\boldsymbol{\mu^l(\tau)}\right)\right)^{\boldsymbol{\lambda}}}.\sqrt{\boldsymbol{1} - \left(\boldsymbol{1} - \left(\boldsymbol{\mu^u(\tau)}\right)\right)^{\boldsymbol{\lambda}}}, \left[\left(\boldsymbol{v^l(\tau)}\right), \left(\boldsymbol{v^u(\tau)}\right)\right], \left[\left(\boldsymbol{\pi^l(\tau)}\right), \left(\boldsymbol{\pi^u(\tau)}\right)\right]\right), \tag{13}$$

$$\boldsymbol{\alpha^\lambda} = \begin{Bmatrix} \left[\left(\boldsymbol{\mu^l(\tau)}\right)^{\boldsymbol{\lambda}}, \left(\boldsymbol{\mu^u(\tau)}\right)^{\boldsymbol{\lambda}}\right], \left[\sqrt{\boldsymbol{1} - \left(\boldsymbol{1} - \left(\boldsymbol{v^l(\tau)}\right)\right)^{\boldsymbol{\lambda}}}, \sqrt{(\boldsymbol{1} -)^{\boldsymbol{\lambda}}}\right], \\ \left[\sqrt{\boldsymbol{1} - \left(\boldsymbol{1} - \left(\boldsymbol{\pi^l(\tau)}\right)\right)^{\boldsymbol{\lambda}}}, \sqrt{\left(\boldsymbol{1} - \left(\boldsymbol{\pi^u(\tau)}\right)\right)^{\boldsymbol{\lambda}}}\right] \end{Bmatrix}. \tag{14}$$

Definition 6. The interval-valued picture weighted arithmetic mean (IVPWAM) is defined as:

$$\boldsymbol{IVPWAM(\alpha_1, \alpha_2, \ldots, \alpha_\eta) = \omega_1.\alpha_1 \oplus \omega_2.\alpha_2 \oplus \ldots \oplus \omega_2.\alpha_\eta} \tag{15}$$

$$= \left\{ \begin{array}{c} \left[\left(1-\prod_{j=1}^{\eta}\left(1-\left(\mu_j^l(\tau)\right)\right)^{\omega_j}\right)^{\frac{1}{2}}, \left(1-\prod_{j=1}^{\eta}\left(1-\left(\mu_j^u(\tau)\right)\right)^{\omega_j}\right)^{\frac{1}{2}}\right], \left[\prod_{j=1}^{\eta}\left(v_j^l(\tau)\right)^{\omega_j}, \prod_{j=1}^{\eta}\left(v_j^u(\tau)\right)^{\omega_j}\right], \\ \left[\begin{array}{c}\left(1-\prod_{j=1}^{\eta}\left(1-\left(\mu_j^l(\tau)\right)\right)^{\omega_j}-\prod_{j=1}^{\eta}\left(1-\left(\mu_j^l(\tau)\right)-\left(\pi_j^l(\tau)\right)\right)^{\omega_j}\right)^{\frac{1}{2}}, \\ \left(1-\prod_{j=1}^{\eta}\left(1-\left(\mu_j^u(\tau)\right)\right)^{\omega_j}-\prod_{j=1}^{\eta}\left(1-\left(\mu_j^u(\tau)\right)-\left(\pi_j^u(\tau)\right)\right)^{\omega_j}\right)^{\frac{1}{2}}\end{array}\right] \end{array} \right\}.$$

Definition 7. The interval-valued picture weighted arithmetic mean (IVPWAM) is defined as:

$$\boldsymbol{IVPWAM_w}(\alpha_1, \alpha_2, \ldots, \alpha_\eta) = \omega_1.\alpha_1 \oplus \omega_2.\alpha_2 \oplus \ldots \oplus \omega_2.\alpha_\eta$$
$$= \left\{ \begin{array}{c} \left[\prod_{j=1}^{\eta}\mu_j^l(\tau)^{\omega_j}, \prod_{j=1}^{\eta}\mu_j^U(\tau)^{\omega_j}\right], \left[\left(1-\prod_{j=1}^{\eta}\left(1-\left(v_j^l(\tau)\right)\right)^{\omega_j}\right)^{\frac{1}{2}}, \left(1-\prod_{j=1}^{\eta}\left(1-\left(v_j^u(\tau)\right)\right)^{\omega_j}\right)^{\frac{1}{2}}\right], \\ \left[\begin{array}{c}\left(\prod_{j=1}^{\eta}\left(1-\left(v_j^l(\tau)\right)\right)^{\omega_j}-\prod_{j=1}^{\eta}\left(1-\left(v_j^l(\tau)\right)-\left(\pi_j^l(\tau)\right)\right)^{\omega_j}\right)^{\frac{1}{2}}, \\ \left(\prod_{j=1}^{\eta}\left(1-\left(v_j^u(\tau)\right)\right)^{\omega_j}-\prod_{j=1}^{\eta}\left(1-\left(v_j^u(\tau)\right)-\left(\pi_j^u(\tau)\right)\right)^{\omega_j}\right)^{\frac{1}{2}}\end{array}\right] \end{array} \right\}. \tag{16}$$

3. Sugeno-Weber Triangular Norm Operation using Interval-Valued Picture Fuzzy Data

Definition 8. For three IVPFSs $\theta = \begin{pmatrix}[\mu^l(\tau), \mu^u(\tau)], \\ [\pi^l(\tau), \pi^u(\tau)], \\ [v^l(\tau), v^u(\tau)]\end{pmatrix}$, $\theta_1 = \begin{pmatrix}[\mu_1^l(\tau), \mu_1^u(\tau)], \\ [\pi_1^l(\tau), \pi_1^u(\tau)], \\ [v_1^l(\tau), v_1^u(\tau)]\end{pmatrix}$, and $\theta_2 = \begin{pmatrix}[\mu_2^l(\tau), \mu_2^u(\tau)], \\ [\pi_2^l(\tau), \pi_2^u(\tau)], \\ [v_2^l(\tau), v_2^u(\tau)]\end{pmatrix}$ Sugeno-Weber TNs fundamental operation is provided by:

$$\boldsymbol{\theta_1 \oplus \theta_2} = \left(\begin{array}{c} \left[\begin{array}{c}\left(\sqrt{\left(\mu_1^l(\tau)\right)+\left(\mu_2^l(\tau)\right)-\frac{\mathbb{R}}{1+\mathbb{R}}\left(\mu_1^l(\tau)\right).\left(\mu_2^l(\tau)\right)}\right), \\ \left(\sqrt{\left(\mu_1^u(\tau)\right)+\left(\mu_2^u(\tau)\right)-\frac{\mathbb{R}}{1+\mathbb{R}}\left(\mu_1^u(\tau)\right).\left(\mu_2^u(\tau)\right)}\right)\end{array}\right], \\ \left[\begin{array}{c}\left(\sqrt{\frac{\left(\pi_1^l(\tau)\right)+\left(\pi_2^l(\tau)\right)-1+\mathbb{R}\left(\pi_1^l(\tau)\right).\left(\pi_2^l(\tau)\right)}{1+\mathbb{R}}}\right), \\ \left(\sqrt{\frac{\left(\pi_1^u(\tau)\right)+\left(\pi_2^u(\tau)\right)-1+\mathbb{R}\left(\pi_1^u(\tau)\right).\left(\pi_2^u(\tau)\right)}{1+\mathbb{R}}}\right)\end{array}\right], \\ \left[\begin{array}{c}\left(\sqrt{\frac{\left(v_1^l(\tau)\right)+\left(v_2^l(\tau)\right)-1+\mathbb{R}\left(v_1^l(\tau)\right).\left(v_2^l(\tau)\right)}{1+\mathbb{R}}}\right), \\ \left(\sqrt{\frac{\left(v_1^u(\tau)\right)+\left(v_2^u(\tau)\right)-1+\mathbb{R}\left(v_1^u(\tau)\right).\left(v_2^u(\tau)\right)}{1+\mathbb{R}}}\right)\end{array}\right] \end{array} \right), \tag{17}$$

$$\boldsymbol{\theta_1}\otimes\boldsymbol{\theta_2}=\left(\begin{array}{c}\left[\begin{array}{c}\left(\sqrt{\frac{(\mu_1^l(\tau))+(\mu_2^l(\tau))-1+\mathbb{R}(\mu_1^l(\tau)).(\mu_2^l(\tau))}{1+\mathbb{R}}}\right),\\ \left(\sqrt{\frac{(\mu_1^l(\tau))+(\mu_2^l(\tau))-1+\mathbb{R}(\mu_1^u(\tau)).(\mu_2^u(\tau))}{1+\mathbb{R}}}\right)\end{array}\right],\\ \left[\begin{array}{c}\left(\sqrt{(\pi_1^l(\tau))+(\pi_2^l(\tau))-\frac{\mathbb{R}}{1+\mathbb{R}}(\pi_1^l(\tau)).(\pi_2^l(\tau))}\right),\\ \left(\sqrt{(\pi_1^l(\tau))+(\pi_2^l(\tau))-\frac{\mathbb{R}}{1+\mathbb{R}}(\pi_1^l(\tau)).(\pi_2^l(\tau))}\right)\end{array}\right],\\ \left[\begin{array}{c}\left(\sqrt{(v_1^l(\tau))+(v_2^l(\tau))-\frac{\mathbb{R}}{1+\mathbb{R}}(v_1^l(\tau)).(v_2^l(\tau))}\right),\\ \left(\sqrt{(v_1^l(\tau))+(v_2^l(\tau))-\frac{\mathbb{R}}{1+\mathbb{R}}(v_1^l(\tau)).(v_2^l(\tau))}\right)\end{array}\right]\end{array}\right), \tag{18}$$

$$\Delta\boldsymbol{\theta}=\left(\begin{array}{c}\left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\left(1-(\mu^l(\tau))\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)^{\Delta}}, \sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\left(1-(\mu^u(\tau))\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)^{\Delta}}\right],\\ \left[\sqrt{(1+\mathbb{R})\left(1-\left(\frac{1+\mathbb{R}(\pi^l(\tau))}{1+\mathbb{R}}\right)^{\Delta}-1\right)\frac{1}{\mathbb{R}}}, \sqrt{(1+\mathbb{R})\left(1-\left(\frac{\mathbb{R}(\pi^u(\tau))+1}{1+\mathbb{R}}\right)^{\Delta}-1\right)\frac{1}{\mathbb{R}}}\right],\\ \left[\sqrt{(1+\mathbb{R})\left(1-\left(\frac{1+\mathbb{R}(v^l(\tau))}{1+\mathbb{R}}\right)^{\Delta}-1\right)\frac{1}{\mathbb{R}}}, \sqrt{(1+\mathbb{R})\left(1-\left(\frac{1+\mathbb{R}(v^u(\tau))}{1+\mathbb{R}}\right)^{\Delta}-1\right)\frac{1}{\mathbb{R}}}\right]\end{array}\right), \tag{19}$$

$$\boldsymbol{\theta}^{\Delta}=\left(\begin{array}{c}\left[\sqrt{\frac{1}{\mathbb{R}}(1+\mathbb{R})\left(1-\left(\frac{\mathbb{R}(\mu^l(\tau))+1}{1+\mathbb{R}}\right)^{\Delta}-1\right)}, \sqrt{\frac{1}{\mathbb{R}}(1+\mathbb{R})\left(1-\left(\frac{\mathbb{R}(\mu^u(\tau))+1}{1+\mathbb{R}}\right)^{\Delta}-1\right)}\right],\\ \left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\left(1-(\pi^l(\tau))\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)^{\Delta}}, \sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\left(1-(\pi^u(\tau))\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)^{\Delta}}\right],\\ \left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\left(1-(v^l(\tau))\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)^{\Delta}}, \sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\left(1-(v^u(\tau))\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)^{\Delta}}\right]\end{array}\right). \tag{20}$$

Definition 9. Consider a class of IVPFSs $\Delta_\iota = ([\mu_\iota^l(\tau),\mu_\iota^u(\tau)],[\pi_\iota^l(\tau),\pi_\iota^u(\tau)],[v_\iota^l(\tau),v_\iota^u(\tau)]), \iota = 1,2,\ldots,\eta$. The following describes the IVPFSWPA operator:

$$\boldsymbol{IVPFSWPA}(\Delta_1,\Delta_2,\ldots,\Delta_\eta)=\oplus_{\iota=1}^{\eta}\boldsymbol{\vartheta_\iota}\Delta_\iota, \tag{21}$$

where $\vartheta_\iota=\frac{(1+\mathbb{L}(\Delta_\iota))}{\sum_{\iota=1}^{\eta}(1+\mathbb{L}(\Delta_\iota))}$ and $A(\Delta_\iota)=\sum_{\iota=1}^{\eta} supp(\Delta_\iota,\Delta_\tau)$.

Theorem 1. Consider a class of IVSFSs $\Delta_\iota = ([\mu_\iota^l(\tau),\mu_\iota^u(\tau)],[\pi_\iota^l(\tau),\pi_\iota^u(\tau)],[v_\iota^l(\tau),v_\iota^u(\tau)]), \iota = 1,2,\ldots,\eta$. We have the following equation:

$$\boldsymbol{IVPFSWPA}(\Delta_{\mathbf{1}},\Delta_{\mathbf{2}},\dots,\Delta_{\eta}) = \left(\begin{array}{c} \left[\begin{array}{c}\left(\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(\mathbf{1}-\prod_{\iota=\mathbf{1}}^{\eta}\left(\mathbf{1}-\left(\boldsymbol{u}_{\iota}^{l}(\boldsymbol{\tau})\right)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)^{\boldsymbol{\vartheta}_{\iota}}}\right), \\ \left(\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(\mathbf{1}-\prod_{\iota=\mathbf{1}}^{\eta}\left(\mathbf{1}-\left(\boldsymbol{u}_{\iota}^{u}(\boldsymbol{\tau})\right)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)^{\boldsymbol{\vartheta}_{\iota}}}\right)\end{array}\right], \\ \left[\begin{array}{c}\sqrt{\frac{1}{\mathbb{R}}\left((\mathbf{1}+\mathbb{R})\prod_{\iota=\mathbf{1}}^{\eta}\left(\frac{\mathbb{R}\left(\boldsymbol{\pi}_{\iota}^{l}(\boldsymbol{\tau})\right)+1}{1+\mathbb{R}}\right)^{\boldsymbol{\vartheta}_{\iota}}-\mathbf{1}\right)}, \\ \sqrt{\frac{1}{\mathbb{R}}\left((\mathbf{1}+\mathbb{R})\prod_{\iota=\mathbf{1}}^{\eta}\left(\frac{\mathbb{R}\left(\boldsymbol{\pi}_{\iota}^{u}(\boldsymbol{\tau})\right)+1}{1+\mathbb{R}}\right)^{\boldsymbol{\vartheta}_{\iota}}-\mathbf{1}\right)}\end{array}\right], \\ \left[\begin{array}{c}\sqrt{\frac{1}{\mathbb{R}}\left((\mathbf{1}+\mathbb{R})\prod_{\iota=\mathbf{1}}^{\eta}\left(\frac{\mathbb{R}\left(\boldsymbol{v}_{\iota}^{l}(\boldsymbol{\tau})\right)+1}{1+\mathbb{R}}\right)^{\boldsymbol{\vartheta}_{\iota}}-\mathbf{1}\right)}, \\ \sqrt{\frac{1}{\mathbb{R}}\left((\mathbf{1}+\mathbb{R})\prod_{\iota=\mathbf{1}}^{\eta}\left(\frac{\mathbb{R}\left(\boldsymbol{v}_{\iota}^{u}(\boldsymbol{\tau})\right)+1}{1+\mathbb{R}}\right)^{\boldsymbol{\vartheta}_{\iota}}-\mathbf{1}\right)}\end{array}\right]\end{array}\right). \tag{22}$$

Proof of Theorem 1 is provided Appendix-1.

Theorem 2. Consider a class of IVPFSs $\Delta_{\iota}=([\mu_{\iota}^{l}(\tau),\mu_{\iota}^{u}(\tau)],[\pi_{\iota}^{l}(\tau),\pi_{\iota}^{u}(\tau)],[v_{\iota}^{l}(\tau),v_{\iota}^{u}(\tau)]),\iota=1,2,\dots,\eta$. Let us assume that they are identical; i.e. $\Delta_{\iota}=\Delta$.Then, we have:

$$\boldsymbol{IVPFSWPA}(\Delta_{\mathbf{1}},\Delta_{\mathbf{2}},\dots,\Delta_{\eta}) = \Delta. \tag{23}$$

Proof of Theorem 2 is provided Appendix-2.

Theorem 3. Consider a class of IVPFSs $\Delta_{\iota}=([\mu_{\iota}^{l}(\tau),\mu_{\iota}^{u}(\tau)],[\pi_{\iota}^{l}(\tau),\pi_{\iota}^{u}(\tau)],[v_{\iota}^{l}(\tau),v_{\iota}^{u}(\tau)]),\iota=1,2,\dots,\eta$, which implies that $\Delta_{\iota}=\Delta$.Then, we obtain:

$$\boldsymbol{IVPFWPOA}\,(\Delta_{\mathbf{1}},\Delta_{\mathbf{2}},\dots,\Delta_{\eta}) = \Delta. \tag{24}$$

Theorem 4. Consider any two sets of IVPFSs $\Delta_{\iota}=([\mu_{\iota}^{l}(\tau),\mu_{\iota}^{u}(\tau)],[\pi_{\iota}^{l}(\tau),\pi_{\iota}^{u}(\tau)],[v_{\iota}^{l}(\tau),v_{\iota}^{u}(\tau)])$ and $\Delta'_{\iota}=([\mu_{\iota}^{\prime l}(\tau),\mu'^{u}_{\iota}(\tau)],[\pi_{\iota}^{\prime l}(\tau),\pi'^{u}_{\iota}(\tau)],[v'^{l}_{\iota}(\tau),v_{\iota}^{\prime u}(\tau)]),\iota=1,2,\dots,\eta$. If $\Delta_{\iota}\leq\Delta'_{\iota}$, then we get:

$$\boldsymbol{IVPFWPOA}(\Delta_{\mathbf{1}},\Delta_{\mathbf{2}},\dots,\Delta_{\eta}) \leq \boldsymbol{q}-\boldsymbol{IVPFWPOA}(\Delta'_{\mathbf{1}},\Delta'_{\mathbf{2}},\dots,\Delta'_{\eta}). \tag{25}$$

Definition 10. For a class of IVPFSs $\Delta_{\iota}=([\mu_{\iota}^{\prime l}(\tau),\mu_{\iota}^{\prime u}(\tau)],[\pi_{\iota}^{\prime l}(\tau),\pi'^{u}_{\iota}(\tau)],[v'^{l}_{\iota}(\tau),v_{\iota}^{\prime u}(\tau)]),\iota=1,2,\dots,\eta$, the following describes the IVPFSWOPA operator:

$$\boldsymbol{IVPFWPOA}(\Delta_{\mathbf{1}}.\Delta_{\mathbf{2}},\dots,\Delta_{\eta}) = \oplus_{\iota=\mathbf{1}}^{\eta}\,\boldsymbol{\vartheta}_{\iota}\Delta_{\boldsymbol{\rho}(\iota)}, \tag{26}$$

where $\vartheta_{\iota}=\frac{(1+\mathbb{L}(\Delta_{\iota}))}{\sum_{\iota=1}^{\eta}(1+\mathbb{L}(\Delta_{\iota}))}$, $\mathbb{L}(\Delta_{\iota})=\sum_{\substack{\iota=1\\ \iota\neq\tau}}^{\eta}supp(\Delta_{\iota},\Delta_{\tau})$, and let $\rho(1),\rho(2),\dots,\rho(\eta)$ be set of the permutation of $\Delta\rho_{(\iota-1)}\geq\Delta_{\iota}$ $(\iota=1,2,\dots,\eta)$.

Definition 11. For a class of IVPFSs $\Delta_{\iota}=([\mu_{\iota}^{l}(\tau),\mu_{\iota}^{u}(\tau)],[\pi_{\iota}^{l}(\tau),\pi_{\iota}^{u}(\tau)],[v_{\iota}^{l}(\tau),v_{\iota}^{u}(\tau)]),\iota=1,2,\dots,\eta$, the following describes the IVPFSWPG operator:

$$\boldsymbol{IVPFSWPG}(\Delta_{\mathbf{1}}.\Delta_{\mathbf{2}},\dots,\Delta_{\eta}) = \otimes_{\iota=1}^{\eta} \Delta_{\iota}^{\vartheta_\iota}, \tag{27}$$

where $\vartheta_\iota = \frac{(1+\mathbb{L}(\Delta_\iota))}{\sum_{\iota=1}^{\eta}(1+\mathbb{L}(\Delta_\iota))}$ and $\mathrm{A}(\Delta_\iota) = \sum_{\substack{\iota=1\\ \iota\neq\tau}}^{\eta} supp(\Delta_\iota,\Delta_\tau)$.

Theorem 5. Assume a class of an IVPFSs $\Delta_\iota = ([\mu_\iota^l(\tau),\mu_\iota^u(\tau)],[\pi_\iota^l(\tau),\pi_\iota^u(\tau)],[v_\iota^l(\tau),v_\iota^u(\tau)]), \iota = 1,2,\dots,\eta$. Since the IVPFSWOPA operator's cumulative value is still a PFV, we obtain the following:

$$\boldsymbol{IVPFSWOPA}(\Delta_{\mathbf{1}},\Delta_{\mathbf{2}},\dots,\Delta_{\eta}) =$$
$$\left(\begin{array}{l}\left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\prod_{\iota=1}^{\eta}\left(1-\left(\mu_{\rho(\iota)}^{l}(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)^{\vartheta_\iota}\right)}, \sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\prod_{\iota=1}^{\eta}\left(1-\left(\mu_{\rho(\iota)}^{u}(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)^{\vartheta_\iota}\right)}\right], \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\prod_{\iota=1}^{\eta}\left(\frac{\mathbb{R}\left(\pi_{\rho(\iota)}^{l}(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_\iota}-1\right)}, \sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\prod_{\iota=1}^{\eta}\left(\frac{\mathbb{R}\left(\pi_{\rho(\iota)}^{u}(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_\iota}-1\right)}\right], \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\prod_{\iota=1}^{\eta}\left(\frac{\mathbb{R}\left(v_{\rho(\iota)}^{l}(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_\iota}-1\right)}, \sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\prod_{\iota=1}^{\eta}\left(\frac{\mathbb{R}\left(v_{\rho(\iota)}^{u}(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_\iota}-1\right)}\right]\end{array}\right), \tag{28}$$

where $\vartheta_\iota = \frac{(1+\mathbb{L}(\Delta_\iota))}{\sum_{\iota=1}^{\eta}(1+\mathbb{L}(\Delta_\iota))}$, $\mathbb{L}(\Delta_\iota) = \sum_{\substack{\iota=1\\ \iota\neq\tau}}^{\eta} supp(\Delta_\iota,\Delta_\tau)$, and $\rho(1),\rho(2),\dots,\rho(\eta)$ be set of the permutation of $\iota = 1,2,\dots,\eta \Delta\rho_{(\iota-1)} \geq \Delta_\iota$.

Theorem 6. Assume a class of IVPFVs $\Delta_\iota = ([\mu_\iota^l(\tau),\mu_\iota^u(\tau)],[\pi_\iota^l(\tau),\pi_\iota^u(\tau)],[v_\iota^l(\tau),v_\iota^u(\tau)]), \iota = 1,2,\dots,\eta$. Since the IVPFSWPG operator's cumulative value is still a PFV, we obtain the following:

$$\mathbf{IVPFSWPG}(\Delta_{\mathbf{1}},\Delta_{\mathbf{2}}\dots,\Delta_{\iota}) = \left(\begin{array}{l}\left[\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\prod_{\iota=1}^{\eta}\left(\frac{\mathbb{R}\left((\mu_\iota^l)(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_\iota}-1\right)}\right], \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\prod_{\iota=1}^{\eta}\left(\frac{\mathbb{R}\left((\mu_\iota^u)(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_\iota}-1\right)}\right] \\ \left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\prod_{\iota=1}^{\eta}\left(1-\left((\pi_\iota^l)(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)^{\vartheta_\iota}\right)}\right], \\ \left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\prod_{\iota=1}^{\eta}\left(1-\left((\pi_\iota^u)(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)^{\vartheta_\iota}\right)}\right] \\ \left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\prod_{\iota=1}^{\eta}\left(1-\left((v_\iota^l)(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)^{\vartheta_\iota}\right)}\right], \\ \left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\prod_{\iota=1}^{\eta}\left(1-\left((v_\iota^u)(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)^{\vartheta_\iota}\right)}\right]\end{array}\right). \tag{29}$$

Theorem 7. Consider a class of IVPFSs $\Delta_\iota = ([\mu_\iota^l(\tau), \mu_\iota^u(\tau)], [\pi_\iota^l(\tau), \pi_\iota^u(\tau)], [v_\iota^l(\tau), v_\iota^u(\tau)]), \iota = 1, 2, \ldots, \eta$, which implies that $\Delta_\iota = \Delta$. Then, we obtain:

$$\boldsymbol{IVPFSWPG}\left(\Delta_{\mathbf{1}}, \Delta_{\mathbf{2}}, \ldots, \Delta_\eta\right) = \Delta. \tag{30}$$

Theorem 8. Consider two sets of IVPFSs Δ_ι and Δ'_ι, $\iota = 1, 2, \ldots, \eta$. If $\Delta_\iota \le \Delta'_\iota$, then we get:

$$\boldsymbol{IVPFSWPG}\left(\Delta_{\mathbf{1}}, \Delta_{\mathbf{2}}, \ldots, \Delta_\eta\right) \le \boldsymbol{q} - \boldsymbol{IVPFSWPG}\left(\Delta'_{\mathbf{1}}, \Delta'_{\mathbf{2}}, \ldots, \Delta'_\eta\right). \tag{31}$$

Definition 12. For a class of a IVPFSs $\Delta_\iota = ([\mu_\iota^l(\tau), \mu_\iota^u(\tau)], [\pi_\iota^l(\tau), \pi_\iota^u(\tau)], [v_\iota^l(\tau), v_\iota^u(\tau)]), \iota = 1, 2, \ldots, \eta$, the following describes the IVPFSWPOG operator:

$$\boldsymbol{IVPFSWPOG}\left(\Delta_{\mathbf{1}}. \Delta_{\mathbf{2}}, \ldots, \Delta_\eta\right) = \oplus_{\boldsymbol{\iota}=\mathbf{1}}^{\eta} \Delta_{\boldsymbol{\rho(\iota)}}^{\boldsymbol{\vartheta_\iota}}, \tag{32}$$

where $\vartheta_\iota = \frac{(1+\mathbb{L}(\Delta_\iota))}{\sum_{\iota=1}^{\eta}(1+\mathbb{L}(\Delta_\iota))}$, $\mathrm{A}(\Delta_\iota) = \sum_{\substack{\iota=1 \\ \iota\neq\tau}}^{\eta} supp(\Delta_\iota, \Delta_\tau)$, and let $\rho(1), \rho(2), \ldots, \rho(\eta)$ be set of the permutation of $\Delta\rho_{(\iota-1)} \ge \Delta_\iota$ $(\iota = 1, 2, \ldots, \eta)$.

Theorem 9. Assume a class of IVPFSs $\Delta_\iota = ([\mu_\iota^l(\tau), \mu_\iota^u(\tau)], [\pi_\iota^l(\tau), \pi_\iota^u(\tau)], [v_\iota^l(\tau), v_\iota^u(\tau)]), \iota = 1, 2, \ldots, \eta$. Since the IVPFSWPOG operator's accumulated value is still a PFV, we obtain the following:

$$\boldsymbol{IVPFSWPOG}(\Delta_{\mathbf{1}}, \Delta_{\mathbf{2}} \ldots, \Delta_\iota) = \begin{pmatrix} \left[\sqrt{\frac{\mathbf{1}}{\mathbb{R}}\left((\mathbf{1}+\mathbb{R})\prod_{\boldsymbol{\iota}=\mathbf{1}}^{\eta}\left(\frac{\mathbb{R}\left((\boldsymbol{\mu}_{\boldsymbol{\iota}}^{\boldsymbol{l}})(\boldsymbol{\tau})\right)+\mathbf{1}}{\mathbf{1}+\mathbb{R}}\right)^{\boldsymbol{\vartheta_\iota}} - \mathbf{1}\right)}\right., \\ \left.\sqrt{\frac{\mathbf{1}}{\mathbb{R}}\left((\mathbf{1}+\mathbb{R})\prod_{\boldsymbol{\iota}=\mathbf{1}}^{\eta}\left(\frac{\mathbb{R}\left((\boldsymbol{\mu}_{\boldsymbol{\iota}}^{\boldsymbol{u}})(\boldsymbol{\tau})\right)+\mathbf{1}}{\mathbf{1}+\mathbb{R}}\right)^{\boldsymbol{\vartheta_\iota}} - \mathbf{1}\right)}\right] \\ \left[\sqrt{\frac{\mathbf{1}+\mathbb{R}}{\mathbb{R}}\left(\mathbf{1}-\prod_{\boldsymbol{\iota}=\mathbf{1}}^{\eta}\left(\mathbf{1}-\left((\boldsymbol{\pi}_{\boldsymbol{\iota}}^{\boldsymbol{l}})(\boldsymbol{\tau})\right)\left(\frac{\mathbb{R}}{\mathbf{1}+\mathbb{R}}\right)\right)^{\boldsymbol{\vartheta_\iota}}\right)}\right., \\ \left.\sqrt{\frac{\mathbf{1}+\mathbb{R}}{\mathbb{R}}\left(\mathbf{1}-\prod_{\boldsymbol{\iota}=\mathbf{1}}^{\eta}\left(\mathbf{1}-\left((\boldsymbol{\pi}_{\boldsymbol{\iota}}^{\boldsymbol{u}})(\boldsymbol{\tau})\right)\left(\frac{\mathbb{R}}{\mathbf{1}+\mathbb{R}}\right)\right)^{\boldsymbol{\vartheta_\iota}}\right)}\right] \\ \left[\sqrt{\frac{\mathbf{1}+\mathbb{R}}{\mathbb{R}}\left(\mathbf{1}-\prod_{\boldsymbol{\iota}=\mathbf{1}}^{\eta}\left(\mathbf{1}-\left((\boldsymbol{v}_{\boldsymbol{\iota}}^{\boldsymbol{l}})(\boldsymbol{\tau})\right)\left(\frac{\mathbb{R}}{\mathbf{1}+\mathbb{R}}\right)\right)^{\boldsymbol{\vartheta_\iota}}\right)}\right., \\ \left.\sqrt{\frac{\mathbf{1}+\mathbb{R}}{\mathbb{R}}\left(\mathbf{1}-\prod_{\boldsymbol{\iota}=\mathbf{1}}^{\eta}\left(\mathbf{1}-\left((\boldsymbol{v}_{\boldsymbol{\iota}}^{\boldsymbol{u}})(\boldsymbol{\tau})\right)\left(\frac{\mathbb{R}}{\mathbf{1}+\mathbb{R}}\right)\right)^{\boldsymbol{\vartheta_\iota}}\right)}\right] \end{pmatrix}. \tag{33}$$

Theorem 10. Consider a class of IVPFVs$\Delta_\iota = ([\mu_\iota^l(\tau), \mu_\iota^u(\tau)], [\pi_\iota^l(\tau), \pi_\iota^u(\tau)], [v_\iota^l(\tau), v_\iota^u(\tau)]), \iota = 1, 2, \ldots, \eta$, which implies that $\Delta_\iota = \Delta$. Then we obtain:

$$\boldsymbol{IVPFSWPOG}\left(\Delta_{\mathbf{1}}, \Delta_{\mathbf{2}}, \ldots, \Delta_\eta\right) = \Delta. \tag{34}$$

Theorem 11. Consider any two sets of IVPFSs $\Delta_\iota = ([\mu_\iota^l(\tau), \mu_\iota^u(\tau)], [\pi_\iota^l(\tau), \pi_\iota^u(\tau)], [v_\iota^l(\tau), v_\iota^u(\tau)])$ and $\Delta'_\iota = ([\mu_\iota'^l(\tau), \mu'^u_\iota(\tau)], [\pi_\iota'^l(\tau), \pi'^u_\iota(\tau)], [v'^l_\iota(\tau), v_\iota'^u(\tau)])$, $\iota = 1, 2, \ldots, \eta$. If $\Delta_\iota \le \Delta'_\iota$, then we get:

$$\boldsymbol{IVPFSWPOG}(\Delta_{\mathbf{1}}, \Delta_{\mathbf{2}}, \dots, \Delta_{\eta}) \leq \boldsymbol{q} - \boldsymbol{IVPFSWPOG}(\Delta'_{\mathbf{1}}, \Delta'_{\mathbf{2}}, \dots, \Delta'_{\eta}). \tag{35}$$

Theorem 12. For any two sets of IVPFSs $\Delta_\iota = ([\mu_\iota^l(\tau), \mu_\iota^u(\tau)], [\pi_\iota^l(\tau), \pi_\iota^u(\tau)], [v_\iota^l(\tau), v_\iota^u(\tau)])$ and $\Delta'_\iota = ([`\mu'^l_\iota(\tau), \mu'^u_\iota(\tau)], [`\pi'^l_\iota(\tau), \pi'^u_\iota(\tau)], [`v'^l_\iota(\tau), v'^u_\iota(\tau)])$ $(\iota = 1, 2, \dots, \eta)$ if $\Delta_\iota \leq \Delta'_\iota$, then:

$$\boldsymbol{IVPFSWPA}(\Delta_{\mathbf{1}}, \Delta_{\mathbf{2}}, \dots, \Delta_{\eta}) \leq \boldsymbol{q} - \boldsymbol{IVPFSWPA}(\Delta'_{\mathbf{1}}, \Delta'_{\mathbf{2}}, \dots, \Delta'_{\eta}). \tag{36}$$

Theorem 13. Consider a class of IVPFSs $\Delta_\iota(\iota = 1, 2, \dots, \eta)$. If $\Delta^- = (min\{\mu_\iota^l, \mu_\iota^l\}, max\{\pi_\iota^l, \pi_\iota^u\}, max\{v_\iota^l, v_\iota^u\})$ and $\Delta^+ = (max\{\mu_\iota^l, \mu_\iota^l\}, min\{\pi_\iota^l, \pi_\iota^u\}, min\{v_\iota^l, v_\iota^u\})$, then we get:

$$\Delta^- \leq \boldsymbol{IVPFSWPA}(\Delta_{\mathbf{1}}, \Delta_{\mathbf{2}}, \dots, \Delta_{\eta}) \leq \Delta^+. \tag{37}$$

Theorem 14. Consider a class of IVPFSs $\Delta_\iota(\iota = 1, 2, \dots, \eta)$. If $\Delta^- = (min\{\mu_\iota^l, \mu_\iota^l\}, max\{\pi_\iota^l, \pi_\iota^u\}, max\{v_\iota^l, v_\iota^u\})$ and $\Delta^+ = (max\{\mu_\iota^l, \mu_\iota^l\}, min\{\pi_\iota^l, \pi_\iota^u\}, min\{v_\iota^l, v_\iota^u\})$, then we get:

$$\Delta^- \leq \boldsymbol{IVPFSWPOA}(\Delta_{\mathbf{1}}, \Delta_{\mathbf{2}}, \dots, \Delta_{\eta}) \leq \Delta^+. \tag{38}$$

Theorem 15. Consider a class of IVPFVs$\Delta_\iota = ([\mu_\iota^l(\tau), \mu_\iota^u(\tau)], [\pi_\iota^l(\tau), \pi_\iota^u(\tau)], [v_\iota^l(\tau), v_\iota^u(\tau)]), \iota = 1, 2, \dots, \eta$. If$\Delta^- = (m\iota\eta\{\mu_\iota^l, \mu_\iota^l\}, max\{\pi_\iota^l, \pi_\iota^u\}, max\{v_\iota^l, v_\iota^u\})$ and$\Delta^+ = (max\{\mu_\iota^l, \mu_\iota^l\}, m\iota\eta\{\pi_\iota^l, \pi_\iota^u\}, m\iota\eta\{v_\iota^l, v_\iota^u\})$, then we get

$$\Delta^- \leq \mathrm{IVPFSWPG}(\Delta_1, \Delta_2, \dots, \Delta_\eta) \leq \Delta^+$$

Theorem 16. Consider a class of IVPFSs $\Delta_\iota(\iota = 1, 2, \dots, \eta)$. If $\Delta^- = (min\{\mu_\iota^l, \mu_\iota^l\}, max\{\pi_\iota^l, \pi_\iota^u\}, max\{v_\iota^l, v_\iota^u\})$ and $\Delta^+ = (max\{\mu_\iota^l, \mu_\iota^l\}, min\{\pi_\iota^l, \pi_\iota^u\}, min\{v_\iota^l, v_\iota^u\})$, then we get:

$$\Delta^- \leq \boldsymbol{IVPFSWPOG}(\Delta_{\mathbf{1}}, \Delta_{\mathbf{2}}, \dots, \Delta_{\eta}) \leq \Delta^+. \tag{39}$$

Theorem 17. Consider a class of IVPFSs $\Delta_\iota(\iota = 1, 2, \dots, \eta)$. If $\Delta^- = (min\{\mu_\iota^l, \mu_\iota^l\}, max\{\pi_\iota^l, \pi_\iota^u\}, max\{v_\iota^l, v_\iota^u\})$ and $\Delta^+ = (max\{\mu_\iota^l, \mu_\iota^l\}, min\{\pi_\iota^l, \pi_\iota^u\}, min\{v_\iota^l, v_\iota^u\})$, then we get:

$$\Delta^- \leq \boldsymbol{IVPFSWPWA}(\Delta_{\mathbf{1}}, \Delta_{\mathbf{2}}, \dots, \Delta_{\eta}) \leq \Delta^+. \tag{40}$$

Theorem 18. Consider a class of IVPFSs $\Delta_\iota(\iota = 1, 2, \dots, \eta)$. If $\Delta^- = (min\{\mu_\iota^l, \mu_\iota^l\}, max\{\pi_\iota^l, \pi_\iota^u\}, max\{v_\iota^l, v_\iota^u\})$ and $\Delta^+ = (max\{\mu_\iota^l, \mu_\iota^l\}, min\{\pi_\iota^l, \pi_\iota^u\}, min\{v_\iota^l, v_\iota^u\})$, then we get:

$$\Delta^- \leq \boldsymbol{IVPFSWPWG}(\Delta_{\mathbf{1}}, \Delta_{\mathbf{2}}, \dots, \Delta_{\eta}) \leq \Delta^+. \tag{41}$$

4. Evaluation of MADM Issues using Interval-Valued Picture Fuzzy Sugeno-Weber Operators

This section uses the structure of the IVPF Sugeno-Weber operators to assess the degree of uncertainty in human opinions. Let $\partial = (\partial_1, \partial_2, \dots, \partial_\eta)$ represents a finite collection of alternatives, and $C = C_1, C_2, \dots, C_m$ represents a finite set of attributes. Assign a weight degree $F = (F_1, F_2, \dots, F_\eta)$

to each attribute so that $\sum_{\iota=1}^{\eta} F_{\iota} = 1$. The following novel algorithm for the MADM approach was suggested:

Step 1 – PFS information provides the decision maker with knowledge about any realistic object, and describes it as a typical matrix for making decisions.

Step 2 – Two categories of qualities, such as advantageous and non-beneficial, were present in the collected data. The conventional decision matrix must be converted into a normalized decision matrix if the provided information comprises more than one type of attribute.

Step 3 – Utilize the following expression to calculate support:

$$\boldsymbol{supp}\left(\Delta_{\iota j}, \Delta_{jk}\right) = \mathbf{1} - \boldsymbol{D}\left(\Delta_{\iota j}, \Delta_{jk}\right), \tag{42}$$

where:

$$\boldsymbol{D}(\Delta_{\iota}, \Delta_{\tau}) = \frac{|(\mu_{\iota}^{l})-(\mu_{k}^{l})|+|(\mu_{j}^{u})-(\mu_{k}^{u})|+|(v_{\iota}^{l})-(v_{j}^{l})|+|(v_{j}^{u})-(v_{k}^{u})|+|(\pi_{\iota}^{l})-(\pi_{j}^{l})|+|(\pi_{j}^{u})-(\pi_{k}^{u})|}{\mathbf{6}}. \tag{43}$$

Step 4 – Determine the weighted support degree:

$$\mathbb{L}(\Delta) = \sum_{\substack{\iota=1 \\ \iota \neq \tau}}^{\eta} \boldsymbol{E}_{\iota}\, \boldsymbol{supp}(\Delta_{\iota}, \Delta_{\tau}), \tag{44}$$

where $E = \left(E_1, E_2, \ldots, E_{\eta}\right)$, $E_{\iota} > 0$, and $\sum_{\substack{\iota=1 \\ \iota \neq \tau}}^{\eta} E_{\iota} = 1$.

Step 5 – Examine the extent of the assistance:

$$\mathfrak{H} = \frac{\boldsymbol{E}_{\iota}(\mathbf{1}+\mathbb{L}(\Delta_{\iota}))}{\sum_{\substack{\iota=1 \\ \iota \neq \tau}}^{\eta} \boldsymbol{E}_{\iota}(\mathbf{1}+\mathbb{L}(\Delta_{\iota}))}. \tag{45}$$

Step 6 – The IVPFSWPA, IVPFSWPG, IVPFSWPWA, and IVPFSWPWG operators are used to combine the supplied data.

Step 7 – The score values of each person are carefully analyzed in order to determine realistically ideal options.

Step 8 – Every score value is rearranged using the ranking and ordering technique.

5. Experimental Case Study

Creating computer systems that can perform tasks that generally require human intelligence is called artificial intelligence (AI). These activities include problem-solving, pattern recognition, language understanding, machine learning (ML), and decision-making. AI machines can perform many tasks such as driving, playing games, image analysis, and speech recognition.

AI has revolutionized entire industries and changed our daily lives at an unprecedented rate. AI is everywhere, from the complex systems that power self-driving cars to virtual assistants like Siri and Alexa that think we need them. Essentially, AI follows human cognitive processes such as learning, problem-solving, and decision-making. It often outperforms humans in these areas. Without explicit instructions, with the help of ML (a set of skills), computers can learn from data and get better results over time. AI has the potential to transform the way we work, communicate, and use technology by driving innovation in many areas. Consider the five options described below:

i. *Machine learning* (A_1) – A branch of AI called ML focuses on developing statistical models and techniques that allow computers to perform tasks without being programmed. The main goal of ML is to enable computers to learn from experience and data and make better predictions, decisions, and actions without explanation.
ii. *Neural networks* (A_2) – Neural networks form the basic concepts of AI and ML, inspired by the structure and function of the human brain. They belong to a class of algorithms used to identify patterns in data and draw conclusions from them. Connections between neurons (sometimes called neurons or neurons) form neural networks.
iii. *Natural language processing* (A_3) – The main goal of natural language processing (NLP) is to enable robots to understand, interpret, and produce human language in a meaningful and useful way. Virtual assistants, catboats, translation assistants, sentiment analysis tools, and data recovery software are just a few of the areas where language processing is useful. Advances in NLP have led to the emergence of advanced language models that can understand and produce human-like text.
iv. *Robotics* (A_4) – A robot is a tool or machine that can operate without constant assistance from humans. Engineering science concepts such as mechanics, electrical machines, computer science, and artificial intelligence are incorporated into robotics.
v. *Computer vision* (A_5) – Computer vision is a branch of computer science and AI whose goal is to enable robots to see and understand ideas in their environment in ways similar to humans. It requires the creation of models and algorithms that allow computers to analyze and make decisions based on visual resources such as images and video. Computer vision is already widely used in industries including medicine (image analysis), automotive (self-driving cars), robotics, surveillance, and augmented reality. Robots can use computer vision to see and interact with their environments to identify objects, navigate through space, and make decisions based on visual cues. Thanks to the combination of computer vision and robotics technology, robots can operate in complex and dynamic environments.

Consider four features when evaluating information about AI:

i. *Adaptability* (G_1),
ii. *Efficiency* (G_2),
iii. *Scalability* (G_3),
iv. *Interpretability* (G_4).

5.1. Results

Decision makers will take into account the previously mentioned features when evaluating information about AI. Theoretically, the decision maker assigns weights to the sample as $(0.10, 0.25, 0.35, 0.30)$. Decision makers use the plan to evaluate pertinent information based on the MADM process algorithm. As a result, the conventional decision matrix is provided in Table 1.

The IVPFSWPA, IVPFSWPG, IVPFSWPWA, and IVPFSWPWG operators are used to combine the supplied data. The results are presented in Table 2. Covered score values corresponding to each alternative are given in Table 3. Finally, rankings of the score values are provided in Table 4. Since $\mathbb{L}_3$ has the greatest score value across all operators, it is evident that it is the finest option among them all.

Table 1
Conventional decision matrix

	G_1	G_2	G_3	G_4
A_1	$\begin{pmatrix}[0.26, 0.34],\\ [0.45, 0.54],\\ [0.53, 0.55]\end{pmatrix}$	$\begin{pmatrix}[0.42, 0.50],\\ [0.24, 0.32],\\ [0.22, 0.32]\end{pmatrix}$	$\begin{pmatrix}[0.14, 0.26],\\ [0.40, 0.53],\\ [0.59, 0.64]\end{pmatrix}$	$\begin{pmatrix}[0.44, 0.45],\\ [0.31, 0.44],\\ [0.13, 0.15]\end{pmatrix}$
A_2	$\begin{pmatrix}[0.32, 0.87],\\ [0.20, 0.33],\\ [0.09, 0.14]\end{pmatrix}$	$\begin{pmatrix}[0.10, 0.14],\\ [0.15, 0.26],\\ [0.43, 0.76]\end{pmatrix}$	$\begin{pmatrix}[0.24, 0.31],\\ [0.46, 0.47],\\ [0.13, 0.15]\end{pmatrix}$	$\begin{pmatrix}[0.21, 0.30],\\ [0.19, 0.20],\\ [0.12, 0.18]\end{pmatrix}$
A_3	$\begin{pmatrix}[0.53, 0.60],\\ [0.31, 0.34],\\ [0.23, 0.34]\end{pmatrix}$	$\begin{pmatrix}[0.35, 0.41],\\ [0.13, 0.17],\\ [0.21, 0.23]\end{pmatrix}$	$\begin{pmatrix}[0.47, 0.57],\\ [0.14, 0.16],\\ [0.20, 0.21]\end{pmatrix}$	$\begin{pmatrix}[0.43, 0.48],\\ [0.45, 0.55],\\ [0.09, 0.15]\end{pmatrix}$
A_4	$\begin{pmatrix}[0.09, 0.14],\\ [0.43, 0.54],\\ [0.42, 0.46]\end{pmatrix}$	$\begin{pmatrix}[0.14, 0.18],\\ [0.42, 0.51],\\ [0.10, 0.16]\end{pmatrix}$	$\begin{pmatrix}[0.11, 0.16],\\ [0.77, 0.80],\\ [0.17, 0.19]\end{pmatrix}$	$\begin{pmatrix}[0.13, 0.16],\\ [0.43, 0.55],\\ [0.16, 0.23]\end{pmatrix}$
A_5	$\begin{pmatrix}[0.19, 0.23],\\ [0.35, 0.40],\\ [0.44, 0.56]\end{pmatrix}$	$\begin{pmatrix}[0.51, 0.56],\\ [0.43, 0.47],\\ [0.12, 0.21]\end{pmatrix}$	$\begin{pmatrix}[0.43, 0.47],\\ [0.15, 0.21],\\ [0.45, 0.51]\end{pmatrix}$	$\begin{pmatrix}[0.34, 0.37],\\ [0.13, 0.18],\\ [0.10, 0.28]\end{pmatrix}$

Table 2
Conventional decision matrix

	IVPFSWPA	IVPFSWPG	IVPFSWPWA	IVPFSWPWG
A_1	$\begin{pmatrix}[0.3376, 0.4029],\\ [0.3625, 0.4684],\\ [0.4067, 0.4518]\end{pmatrix}$	$\begin{pmatrix}[0.3316, 0.3981],\\ [0.3659, 0.4729],\\ [0.4242, 0.4706]\end{pmatrix}$	$\begin{pmatrix}[0.3396, 0.4031],\\ [0.3522, 0.4641],\\ [0.3974, 0.4472]\end{pmatrix}$	$\begin{pmatrix}[0.3326, 0.3978],\\ [0.3551, 0.4684],\\ [0.4173, 0.4698]\end{pmatrix}$
A_2	$\begin{pmatrix}[0.2340, 0.5298],\\ [0.2703, 0.3403],\\ [0.2386, 0.3906]\end{pmatrix}$	$\begin{pmatrix}[0.2322, 0.4664],\\ [0.2779, 0.3447],\\ [0.2473, 0.4400]\end{pmatrix}$	$\begin{pmatrix}[0.2179, 0.3957],\\ [0.3022, 0.3554],\\ [0.2348, 0.3804]\end{pmatrix}$	$\begin{pmatrix}[0.2167, 0.3574],\\ [0.3109, 0.3611],\\ [0.2430, 0.4283]\end{pmatrix}$
A_3	$\begin{pmatrix}[0.4496, 0.5222],\\ [0.2967, 0.3520],\\ [0.4773, 0.2377]\end{pmatrix}$	$\begin{pmatrix}[0.4472, 0.5183],\\ [0.3040, 0.3653],\\ [0.5566, 0.2396]\end{pmatrix}$	$\begin{pmatrix}[0.4382, 0.5160],\\ [0.2812, 0.3382],\\ [0.4837, 0.2168]\end{pmatrix}$	$\begin{pmatrix}[0.4364, 0.5124],\\ [0.2898, 0.3535],\\ [0.5643, 0.2180]\end{pmatrix}$
A_4	$\begin{pmatrix}[0.1182, 0.1693],\\ [0.5341, 0.6160],\\ [0.2424, 0.2834]\end{pmatrix}$	$\begin{pmatrix}[0.1182, 0.1692],\\ [0.5551, 0.6311],\\ [0.2490, 0.2904]\end{pmatrix}$	$\begin{pmatrix}[0.1225, 0.1744],\\ [0.5598, 0.6366],\\ [0.1919, 0.2368]\end{pmatrix}$	$\begin{pmatrix}[0.1225, 0.1744],\\ [0.5826, 0.6531],\\ [0.1959, 0.2408]\end{pmatrix}$
A_5	$\begin{pmatrix}[0.3895, 0.4297],\\ [0.2944, 0.3394],\\ [0.3170, 0.4093]\end{pmatrix}$	$\begin{pmatrix}[0.3831, 0.4218],\\ [0.3004, 0.3458],\\ [0.3266, 0.4206]\end{pmatrix}$	$\begin{pmatrix}[0.4118, 0.4519],\\ [0.2628, 0.3084],\\ [0.3046, 0.3899]\end{pmatrix}$	$\begin{pmatrix}[0.4077, 0.4467],\\ [0.2698, 0.3156],\\ [0.3148, 0.3998]\end{pmatrix}$

Table 3
Covered score values corresponding to each alternative

	IVPFSWPA	IVPFSWPG	IVPFSWPWA	IVPFSWPWG
A_1	0.5187	0.5032	0.5268	0.5095
A_2	0.6457	0.6069	0.5955	0.5684
A_3	0.6595	0.6252	0.6613	0.6260
A_4	0.4129	0.3966	0.4113	0.3944
A_5	0.6222	0.6104	0.6549	0.6448

Table 4
Ranking of the score values

Operator	Ranking
IVPFSWPA	$\mathbb{L}_3 \succ \mathbb{L}_2 \succ \mathbb{L}_5 \succ \mathbb{L}_1 \succ \mathbb{L}_4$
IVPFSWPG	$\mathbb{L}_3 \succ \mathbb{L}_2 \succ \mathbb{L}_5 \succ \mathbb{L}_1 \succ \mathbb{L}_4$
IVPFSWPWA	$\mathbb{L}_3 \succ \mathbb{L}_5 \succ \mathbb{L}_2 \succ \mathbb{L}_1 \succ \mathbb{L}_4$
IVPFSWPWG	$\mathbb{L}_3 \succ \mathbb{L}_5 \succ \mathbb{L}_2 \succ \mathbb{L}_1 \succ \mathbb{L}_4$

5.2. Sensitivity Analysis

This part aims to explain an investigation of the behavior of the different ranking orders while varying the parameter $\mathbb{R}$ from 1 to 100. The ranking orders of the alternatives stay the same across the whole range of values examined, as presented in Table 5. This implies that inside this range, the alternatives' ranking orders are constant and unaffected by variations in the parameter $\mathbb{R}$. Since $\mathbb{L}_3$ has the highest score value across all operators, it is the best option available. Thus, we conclude that the best AI function is NLP ($\mathbb{L}_3$).

Table 5
Results of the sensitivity analysis

Parametric value	IVPFSWPWA	IVPFSWPWG
$\mathbb{R}=1$	$\mathbb{L}_3 \succ \mathbb{L}_5 \succ \mathbb{L}_2 \succ \mathbb{L}_1 \succ \mathbb{L}_4$	$\mathbb{L}_3 \succ \mathbb{L}_5 \succ \mathbb{L}_2 \succ \mathbb{L}_1 \succ \mathbb{L}_4$
$\mathbb{R}=5$	$\mathbb{L}_3 \succ \mathbb{L}_5 \succ \mathbb{L}_2 \succ \mathbb{L}_1 \succ \mathbb{L}_4$	$\mathbb{L}_3 \succ \mathbb{L}_5 \succ \mathbb{L}_2 \succ \mathbb{L}_1 \succ \mathbb{L}_4$
$\mathbb{R}=20$	$\mathbb{L}_3 \succ \mathbb{L}_2 \succ \mathbb{L}_5 \succ \mathbb{L}_1 \succ \mathbb{L}_4$	$\mathbb{L}_3 \succ \mathbb{L}_2 \succ \mathbb{L}_5 \succ \mathbb{L}_1 \succ \mathbb{L}_4$
$\mathbb{R}=35$	$\mathbb{L}_3 \succ \mathbb{L}_5 \succ \mathbb{L}_2 \succ \mathbb{L}_1 \succ \mathbb{L}_4$	$\mathbb{L}_3 \succ \mathbb{L}_5 \succ \mathbb{L}_2 \succ \mathbb{L}_1 \succ \mathbb{L}_4$
$\mathbb{R}=55$	$\mathbb{L}_3 \succ \mathbb{L}_5 \succ \mathbb{L}_2 \succ \mathbb{L}_1 \succ \mathbb{L}_4$	$\mathbb{L}_3 \succ \mathbb{L}_5 \succ \mathbb{L}_2 \succ \mathbb{L}_1 \succ \mathbb{L}_4$
$\mathbb{R}=70$	$\mathbb{L}_3 \succ \mathbb{L}_5 \succ \mathbb{L}_2 \succ \mathbb{L}_1 \succ \mathbb{L}_4$	$\mathbb{L}_3 \succ \mathbb{L}_5 \succ \mathbb{L}_2 \succ \mathbb{L}_1 \succ \mathbb{L}_4$
$\mathbb{R}=80$	$\mathbb{L}_3 \succ \mathbb{L}_2 \succ \mathbb{L}_5 \succ \mathbb{L}_1 \succ \mathbb{L}_4$	$\mathbb{L}_3 \succ \mathbb{L}_2 \succ \mathbb{L}_5 \succ \mathbb{L}_1 \succ \mathbb{L}_4$
$\mathbb{R}=90$	$\mathbb{L}_3 \succ \mathbb{L}_5 \succ \mathbb{L}_2 \succ \mathbb{L}_1 \succ \mathbb{L}_4$	$\mathbb{L}_3 \succ \mathbb{L}_5 \succ \mathbb{L}_2 \succ \mathbb{L}_1 \succ \mathbb{L}_4$
$\mathbb{R}=100$	$\mathbb{L}_3 \succ \mathbb{L}_5 \succ \mathbb{L}_2 \succ \mathbb{L}_1 \succ \mathbb{L}_4$	$\mathbb{L}_3 \succ \mathbb{L}_5 \succ \mathbb{L}_2 \succ \mathbb{L}_1 \succ \mathbb{L}_4$

5.3. Comparative Analysis

We compare a subset of contemporary operators with the pioneering operators in this part. The results are shown in Table 6. With the use of various demonstrative particle examples, we were able to show the great improvement of the derived work over the current operators by contrasting the unique and current techniques to improve the value and capacity of the diagnosed operators.

Table 6
Results of the comparative analysis

Methods	Ranking information
IVPFSWPA	$\mathbb{L}_3 \succ \mathbb{L}_2 \succ \mathbb{L}_5 \succ \mathbb{L}_1 \succ \mathbb{L}_4$
IVPFSWPG	$\mathbb{L}_3 \succ \mathbb{L}_2 \succ \mathbb{L}_5 \succ \mathbb{L}_1 \succ \mathbb{L}_4$
IVPFSWPWA	$\mathbb{L}_3 \succ \mathbb{L}_5 \succ \mathbb{L}_2 \succ \mathbb{L}_1 \succ \mathbb{L}_4$
IVPFSWPWG	$\mathbb{L}_3 \succ \mathbb{L}_5 \succ \mathbb{L}_2 \succ \mathbb{L}_1 \succ \mathbb{L}_4$
Sarfraz [10]	$\mathbb{L}_3 \succ \mathbb{L}_2 \succ \mathbb{L}_5 \succ \mathbb{L}_1 \succ \mathbb{L}_4$
Arya & Kumar [20]	$\mathbb{L}_3 \succ \mathbb{L}_5 \succ \mathbb{L}_2 \succ \mathbb{L}_1 \succ \mathbb{L}_4$

6. Conclusion

In this work, we first introduced the fundamental functions of IVPFS using the Sugeno-Weber TNs. Sugeno-Weber AOs were then used to aggregate IVPFS data, which contained four distinct types of AOs; i.e. IVPFSWPA, IVPFSWPG, IVPFSWPWA, and IVPFSWPWG. The article highlights notable properties of these AOs. Application scenarios of those operators in resolving MADM issues were illustrated using a real-world example involving the prioritization and evaluation of AI models.

The effectiveness of the derived approaches, which were shown to be relevant in the impact study, was validated by a comparison analysis that contrasted the outcomes with those of existing approaches and created aggregation procedures. In the future, we will primarily concentrate on theoretical and applied research on complex spherical fuzzy, and complex bipolar soft sets.

Appendix-1: Proof of Theorem 1

Theorem 1 can be demonstrated in the following way. The induction procedure $\eta = 2$ can be used to prove the aforementioned expression. We are able to write:

$$\boldsymbol{\vartheta_1}\Delta_1 = \left(\begin{array}{c} \left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\left(1-\left(\mu_1^l(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)\right)^{\vartheta_1}}\right], \left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\left(1-\left(\mu_1^u(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)\right)^{\vartheta_1}}\right] \\ \left[\sqrt{\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(\pi_1^l(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_1}-1\right)\frac{1}{\mathbb{R}}}\right], \left[\sqrt{\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(\pi_1^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_1}-1\right)\frac{1}{\mathbb{R}}}\right] \\ \left[\sqrt{\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(v_1^l(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_1}-1\right)\frac{1}{\mathbb{R}}}\right], \left[\sqrt{\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(v_1^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_1}-1\right)\frac{1}{\mathbb{R}}}\right] \end{array}\right),$$

$$\boldsymbol{\vartheta_2}\Delta_2 = \left(\begin{array}{c} \left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\left(1-\left(\mu_2^l(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)\right)^{\vartheta_2}}\right], \left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\left(1-\left(\mu_2^u(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)\right)^{\vartheta_2}}\right] \\ \left[\sqrt{\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(\pi_2^l(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_2}-1\right)\frac{1}{\mathbb{R}}}\right], \left[\sqrt{\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(\pi_2^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_2}-1\right)\frac{1}{\mathbb{R}}}\right] \\ \left[\sqrt{\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(v_2^l(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_2}-1\right)\frac{1}{\mathbb{R}}}\right], \left[\sqrt{\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(v_2^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_2}-1\right)\frac{1}{\mathbb{R}}}\right] \end{array}\right).$$

$$\boldsymbol{IVPFSWPA}(\Delta_1, \Delta_2) = \left(\begin{array}{c} \left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\left(1-\left(\mu_1^l(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)\right)^{\vartheta_1}}, \sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\left(1-\left(\mu_1^u(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)\right)^{\vartheta_1}}\right] \oplus \\ \left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\left(1-\left(\mu_2^l(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)\right)^{\vartheta_1}}, \sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\left(1-\left(\mu_2^u(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)\right)^{\vartheta_1}}\right], \\ \left[\sqrt{\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(\pi_1^l(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_1}-1\right)\frac{1}{\mathbb{R}}}, \sqrt{\left((1+\mathbb{R}))\left(\frac{\mathbb{R}\left(\pi_1^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_1}-1\right)\frac{1}{\mathbb{R}}}\right] \\ \left[\sqrt{\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(\pi_2^l(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_1}-1\right)\frac{1}{\mathbb{R}}}, \sqrt{\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(\pi_2^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_1}-1\right)\frac{1}{\mathbb{R}}}\right] \oplus \\ \left[\sqrt{\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(v_1^l(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_1}-1\right)\frac{1}{\mathbb{R}}}, \sqrt{\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(v_1^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_1}-1\right)\frac{1}{\mathbb{R}}}\right] \oplus \\ \left[\sqrt{\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(v_2^l(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_1}-1\right)\frac{1}{\mathbb{R}}}. \sqrt{\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(v_2^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_1}-1\right)\frac{1}{\mathbb{R}}}\right] \end{array}\right)$$

$$
\boldsymbol{IVPFSWPA}(\Delta_1,\Delta_2)=\left(\begin{array}{c}
\left[\sqrt{\begin{array}{c}\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\left(1-\left(\mu_1^l(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)\right)^{\vartheta_1}+\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\left(1-\left(\mu_2^l(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)\right)^{\vartheta_2}\\ -\frac{\mathbb{R}}{1+\mathbb{R}}\left(\left(\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\left(1-\left(\mu_1^l(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)\right)^{\vartheta_1}\right).\left(\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\left(1-\left(\mu_2^l(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)\right)^{\vartheta_2}\right)\right)\end{array}}\right],\\
\left[\sqrt{\begin{array}{c}\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\left(1-\left(\mu_1^u(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)\right)^{\vartheta_1}+\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\left(1-\left(\mu_2^u(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)\right)^{\vartheta_2}\\ -\frac{\mathbb{R}}{1+\mathbb{R}}\left(\left(\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\left(1-\left(\mu_1^u(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)\right)^{\vartheta_1}\right).\left(\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\left(1-\left(\mu_2^u(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)\right)^{\vartheta_2}\right)\right)\end{array}}\right]\\
\left[\sqrt{\begin{array}{c}\frac{\mathbb{R}}{1+\mathbb{R}}\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(\pi_1^l(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_1}-1\right)\frac{1}{\mathbb{R}}+\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(\pi_2^l(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_2}-1\right)\frac{1}{\mathbb{R}}-1\\ +\mathbb{R}\left(\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(\pi_1^l(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_1}-1\right)\frac{1}{\mathbb{R}}\right).\left(\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(\pi_2^l(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_2}-1\right)\frac{1}{\mathbb{R}}\right)\end{array}}\right],\\
\left[\sqrt{\begin{array}{c}\frac{\mathbb{R}}{1+\mathbb{R}}\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(\pi_1^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_1}-1\right)\frac{1}{\mathbb{R}}+\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(\pi_2^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_2}-1\right)\frac{1}{\mathbb{R}}-1\\ +\mathbb{R}\left(\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(\pi_1^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_1}-1\right)\frac{1}{\mathbb{R}}\right).\left(\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(\pi_2^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_2}-1\right)\frac{1}{\mathbb{R}}\right)\end{array}}\right]\\
\left[\sqrt{\begin{array}{c}\frac{\mathbb{R}}{1+\mathbb{R}}\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(v_1^l(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_1}-1\right)\frac{1}{\mathbb{R}}+\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(v_2^l(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_2}-1\right)\frac{1}{\mathbb{R}}-1\\ +\mathbb{R}\left(\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(v_1^l(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_1}-1\right)\frac{1}{\mathbb{R}}\right).\left(\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(v_2^l(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_2}-1\right)\frac{1}{\mathbb{R}}\right)\end{array}}\right],\\
\left[\sqrt{\begin{array}{c}\frac{\mathbb{R}}{1+\mathbb{R}}\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(v_1^u(\tau)\right)^2+1}{1+\mathbb{R}}\right)^{\vartheta_1}-1\right)\frac{1}{\mathbb{R}}+\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(v_2^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_2}-1\right)\frac{1}{\mathbb{R}}-1\\ +\mathbb{R}\left(\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(v_1^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_1}-1\right)\frac{1}{\mathbb{R}}\right).\left(\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(v_2^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_2}-1\right)\frac{1}{\mathbb{R}}\right)\end{array}}\right]
\end{array}\right)
$$

$$\boldsymbol{IVPFSWPA}(\Delta_1,\Delta_2)=\begin{pmatrix}\left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\prod_{\iota=1}^{2}\left(1-\left(\mu_\iota^l(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)^{\vartheta_\iota}\right)},\right. \\ \left.\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\prod_{\iota=1}^{2}\left(1-\left(\mu_\iota^u(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)^{\vartheta_\iota}\right)}\right] \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\prod_{\iota=1}^{2}\left(\frac{\mathbb{R}\left(\pi_\iota^l(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_\iota}-1\right)},\right. \\ \left.\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\prod_{\iota=1}^{2}\left(\frac{\mathbb{R}\left(\pi_\iota^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_\iota}-1\right)}\right] \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\prod_{\iota=1}^{2}\left(\frac{\mathbb{R}\left(\nu_\iota^l(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_\iota}-1\right)},\right. \\ \left.\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\prod_{\iota=1}^{2}\left(\frac{\mathbb{R}\left(\nu_\iota^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_\iota}-1\right)}\right]\end{pmatrix}$$

Assume that the aforementioned statement is accurate for $\eta = k$:

$$\boldsymbol{IVPFSWPA}(\Delta_1,\Delta_2,\dots,\Delta_k)=\begin{pmatrix}\left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\prod_{\iota=1}^{k}\left(1-\left(\mu_\iota^l(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)^{\vartheta_\iota}\right)},\right. \\ \left.\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\prod_{\iota=1}^{k}\left(1-\left(\mu_\iota^u(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)^{\vartheta_\iota}\right)}\right] \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\prod_{\iota=1}^{k}\left(\frac{\mathbb{R}\left(\pi_\iota^l(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_\iota}-1\right)},\right. \\ \left.\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\prod_{\iota=1}^{k}\left(\frac{\mathbb{R}\left(\pi_\iota^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_\iota}-1\right)}\right] \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\prod_{\iota=1}^{k}\left(\frac{\mathbb{R}\left(\nu_\iota^l(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_\iota}-1\right)},\right. \\ \left.\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\prod_{\iota=1}^{k}\left(\frac{\mathbb{R}\left(\nu_\iota^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_\iota}-1\right)}\right]\end{pmatrix}$$

We can prove the expression for $\eta = k+1$:

$$\boldsymbol{IVPFSWPA}(\Delta_1,\Delta_2,\dots,\Delta_k)=\oplus_{\iota=1}^{k}\boldsymbol{\vartheta}_k\Delta_k\oplus\boldsymbol{\vartheta}_{k+1}\Delta_{k+1}$$

$$= \begin{pmatrix} \left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\prod_{\iota=1}^{k}\left(1-\left(\mu_\iota^l(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)^{\vartheta_\iota}\right)}\right. \\ \left.\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\prod_{\iota=1}^{k}\left(1-\left(\mu_\iota^u(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)^{\vartheta_\iota}\right)}\right], \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\prod_{\iota=1}^{k}\left(\frac{\mathbb{R}\left(\pi_\iota^l(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_\iota}-1\right)}\right., \\ \left.\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\prod_{\iota=1}^{k}\left(\frac{\mathbb{R}\left(\pi_\iota^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_\iota}-1\right)}\right] \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\prod_{\iota=1}^{k}\left(\frac{\mathbb{R}\left(v_\iota^l(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_\iota}-1\right)}\right., \\ \left.\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\prod_{\iota=1}^{k}\left(\frac{\mathbb{R}\left(v_\iota^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_\iota}-1\right)}\right] \end{pmatrix} \oplus \begin{pmatrix} \left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\left(1-\left(\mu_{k+1}^l(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)^{\vartheta_{k+1}}\right)}\right., \\ \left.\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\left(1-\left(\mu_{k+1}^u(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)^{\vartheta_{k+1}}\right)}\right], \\ \left[\sqrt{\frac{1}{\mathbb{R}}(1+\mathbb{R})\left(\frac{\mathbb{R}\left(\pi_{k+1}^l(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_{k+1}}-1}\right., \\ \left.\sqrt{\frac{1}{\mathbb{R}}(1+\mathbb{R})\left(\frac{\mathbb{R}\left(\pi_{k+1}^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_{k+1}}-1}\right], \\ \left[\sqrt{\frac{1}{\mathbb{R}}(1+\mathbb{R})\left(\frac{\mathbb{R}\left(v_{k+1}^l(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_{k+1}}-1}\right., \\ \left.\sqrt{\frac{1}{\mathbb{R}}(1+\mathbb{R})\left(\frac{\mathbb{R}\left(v_{k+1}^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_{k+1}}-1}\right] \end{pmatrix}.$$

$$\boldsymbol{IVPFSWPA}(\Delta_1, \Delta_2, \dots, \Delta_{k+1}) = \begin{pmatrix} \left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\prod_{\iota=1}^{k+1}\left(1-\left(\mu_\iota^l(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)^{\vartheta_\iota}\right)}\right., \\ \left.\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\prod_{\iota=1}^{k+1}\left(1-\left(\mu_\iota^u(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)^{\vartheta_\iota}\right)}\right] \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\prod_{\iota=1}^{k+1}\left(\frac{\mathbb{R}\left(\pi_\iota^l(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_\iota}-1\right)}\right., \\ \left.\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\prod_{\iota=1}^{k+1}\left(\frac{\mathbb{R}\left(\pi_\iota^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_\iota}-1\right)}\right] \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\prod_{\iota=1}^{k+1}\left(\frac{\mathbb{R}\left(v_\iota^l(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_\iota}-1\right)}\right., \\ \left.\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\prod_{\iota=1}^{k+1}\left(\frac{\mathbb{R}\left(v_\iota^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_\iota}-1\right)}\right] \end{pmatrix}$$

Hence, Eq. (22) is proved.

Appendix-2: Proof of Theorem 2

Theorem 2 can be demonstrated in the following way. Since all IVPFSs $\Delta_\iota = ([\mu_\iota^l(\tau), \mu_\iota^u(\tau)], [\pi_\iota^l(\tau), \pi_\iota^u(\tau)], [v_\iota^l(\tau), v_\iota^u(\tau)]), \iota = 1, 2, \dots, \Pi$ are identical $\Delta_\iota = \Delta$, then we have:

$$\boldsymbol{IVPFSWPA}(\Delta_1, \Delta_2, \ldots, \Delta_\eta) = \begin{pmatrix} \left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\prod_{\iota=1}^{\eta}\left(1-\left(\mu_\iota^l(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)^{\vartheta_\iota}\right)}\right], \\ \left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\prod_{\iota=1}^{\eta}\left(1-\left(\mu_\iota^u(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)^{\vartheta_\iota}\right)}\right] \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\prod_{\iota=1}^{\eta}\left(\frac{\mathbb{R}\left(\pi_\iota^l(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_\iota}-1\right)}\right], \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\prod_{\iota=1}^{\eta}\left(\frac{\mathbb{R}\left(\pi_\iota^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_\iota}-1\right)}\right] \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\prod_{\iota=1}^{\eta}\left(\frac{\mathbb{R}\left(v_\iota^l(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_\iota}-1\right)}\right], \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\prod_{\iota=1}^{\eta}\left(\frac{\mathbb{R}\left(v_\iota^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\vartheta_\iota}-1\right)}\right] \end{pmatrix}$$

$$= \begin{pmatrix} \left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\left(1-\left(\mu^\iota(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)^{\sum_{i=1}^{\eta}\vartheta_\iota}\right)}\right], \\ \left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\left(1-\left(\mu^u(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)^{\sum_{i=1}^{\eta}\vartheta_\iota}\right)}\right] \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(\pi^\iota(\tau)\right)+1}{1+\mathbb{R}}\right)^{\sum_{i=1}^{\eta}\omega_\iota}-1\right)}\right], \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(\pi^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\sum_{i=1}^{\eta}\omega_\iota}-1\right)}\right] \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(v^\iota(\tau)\right)+1}{1+\mathbb{R}}\right)^{\sum_{i=1}^{\eta}\omega_\iota}-1\right)}\right], \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(v^u(\tau)\right)+1}{1+\mathbb{R}}\right)^{\sum_{i=1}^{\eta}\omega_\iota}-1\right)}\right] \end{pmatrix} = \begin{pmatrix} \left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\left(1-\left(\mu^\iota(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)\right)}\right], \\ \left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-\left(1-\left(\mu^u(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)\right)}\right] \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(\pi^\iota(\tau)\right)+1}{1+\mathbb{R}}\right)-1\right)}\right], \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(\pi^u(\tau)\right)+1}{1+\mathbb{R}}\right)-1\right)}\right] \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(v^\iota(\tau)\right)+1}{1+\mathbb{R}}\right)-1\right)}\right], \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left((1+\mathbb{R})\left(\frac{\mathbb{R}\left(v^u(\tau)\right)+1}{1+\mathbb{R}}\right)-1\right)}\right] \end{pmatrix}$$

$$= \begin{pmatrix} \left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-1+\left(\mu^{\iota}(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)}\right], \\ \left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(1-1+\left(\mu^{u}(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)}\right], \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left(\mathbb{R}\left(\pi^{\iota}(\tau)\right)+1-1\right)}\right], \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left(\mathbb{R}\left(\pi^{u}(\tau)\right)+1-1\right)}\right] \\ , \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left(\mathbb{R}\left(v^{\iota}(\tau)\right)+1-1\right)}\right], \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left(\mathbb{R}\left(v^{u}(\tau)\right)+1-1\right)}\right] \end{pmatrix} = \begin{pmatrix} \left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(\left(\mu^{\iota}(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)}\right], \left[\sqrt{\frac{1+\mathbb{R}}{\mathbb{R}}\left(\left(\mu^{u}(\tau)\right)\left(\frac{\mathbb{R}}{1+\mathbb{R}}\right)\right)}\right] \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left(\mathbb{R}\left(\pi^{\iota}(\tau)\right)\right)}\right], \left[\sqrt{\frac{1}{\mathbb{R}}\left(\mathbb{R}\left(\pi^{u}(\tau)\right)\right)}\right] \\ \left[\sqrt{\frac{1}{\mathbb{R}}\left(\mathbb{R}\left(v^{\iota}(\tau)\right)\right)}\right], \left[\sqrt{\frac{1}{\mathbb{R}}\left(\mathbb{R}\left(v^{u}(\tau)\right)\right)}\right] \end{pmatrix}$$

$$= \left(\mu(\tau), \pi(\tau), v(\tau)\right) = \Delta.$$

Hence, Eq. (22) is proved.

Funding

This study did not receive any external financial support.

Conflicts of Interest

The author declares no conflicts of interest.

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